Study Material

Dept. of Applied Mathematics with Oceanology and Computer Programming

Paper No. – MTM 205

Paper Name- Continuum Mechanics

Semester – 2

Topic of Lecturer:

Concept of Elasticity, Hooke's Law, Strain Energy, Existence of Strain Energy Function, Isotropic elastic body, its constitutive equation and basic elastic constants

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Lecturer No.- 03

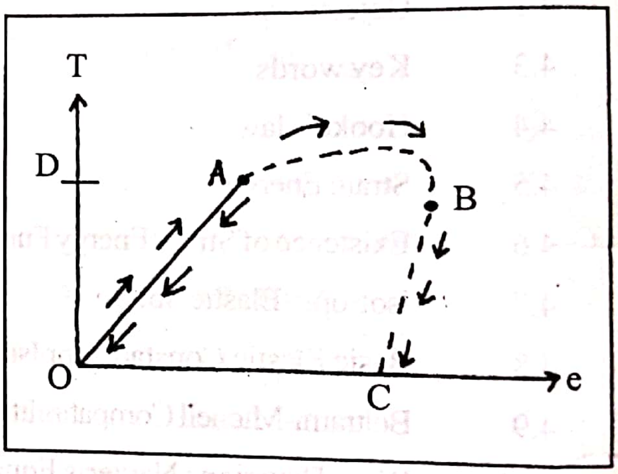
4.1 Introduction :

Elastic Solid : In our universe there are different forms for different materials, and depending on those forms there are different theories of elastic, plastic, viscoelastic bodies, viscous, non-viscous i.e., inviscid fluids etc.

A continuous medium is called *elastic if stress tensor is a continuous function of strain tensor such that stress tensor automatically vanishes when strain tensor becomes zero.* A solid body that consists of this material thus recovers its original shape and size completely whenever all stresses causing the deformation are removed. In this case strain is fully recoverable. *The property by which a continuous body recovers from strain is called elasticity.*

Next we define a linearly elastic solid to be a continuous material (such as metals, concrete, wood) which undergoes very small change of shape when subjected to forces of reasonable magnitude such that every stress component is a linear function of all strain components. It has a natural shape to which it will return whenever all forces causing the deformation are removed provided the forces are not too large. Also it is restricted to the case in which deformation and gradients are small.

To understand the mechanical behaviour of solid, we consider a thin steel rod subjected to a variable tensile stress *T*. *T* will produce an extension *e*. If *T* is plotted as a function of *e*, then adjacent figure will be obtained. If *T* is increased to *D* so that extension lies within *OA*, and *T* is removed then same line *OA* is retraced, so that there is no permanent deformation or extension and the rod returns to its original length. Then rod exhibits elasticity and greatest stress *OD* is called elastic limit of the material. If *T* is increased beyond *D* such that extension goes from *A* to *B* and *T* is removed, the line *BC* is retraced, not the curve *BAO*, so that there is a permanent extension or plastic strain *OC* after the removal of *T*. In this case rod does not fully recover its original length completely. The rod exhibits plasticity.



Thus the behaviour of linear elastic material is subjected to the stress within proportional range *OA*.

4.2 Objectives :

In this module, the students will learn about elastic solid and the general concept of stress-strain relation, wave equation etc.

4.3 Key Words : Elastic solid, Isotrpic media, Hooke's law, Strain energy, Strain energy function, Constitutive equation, Elastic constants, Lame's constant, Bulk modules, Clapeyron's formula, Wave equation.

Hooke's law :

For a linear elastic solid, the strain deformation of a body which gives rise to stresses and the stresses are linear function of infinitesimal strains. So, we can write

$$T_{ij} = B_{ij} + C_{ijkl}e_{kl}$$

Since $T_{ij} = 0$, when all $e_{ij} = 0$, and then $B_{ij} = 0$.

$$\therefore T_{ij} = C_{ijkl}e_{kl}, (i, j, k, l = 1, 2, 3) \;\ldots\ldots\; (1)$$

This relation between stress and strain is known as generalized Hooke's law for linear elastic material.

The coefficients C_{ijkl} are called elastic constants or elastic moduli of the body since they characterise the elastic properties of the body. The elastic solid is said to be elastically non-homogeneous or inhomogeneous if these elastic constants vary from point to point of the medium, and if the elastic constants are the same for all points of the medium then it will be called elastically homogeneous. For an example *mild steel* is homogeneous whereas *reinforced concrete* is non-homogeneous.

Note : We observe that C_{ijkl} form fourth order tensor, known as 'elasticity tensor' and it has $3^4 = 81$ components.

Now

$$\left(C_{ijkl}\right) = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121} \end{pmatrix}$$

Again, since $T_{ij} = T_{ji}$ so from (1) we get $C_{ijkl}e_{kl} = C_{ijkl}e_{kl}$ i.e., $C_{ijkl} = C_{ijkl}$.

Also $C_{ijkl} = \frac{1}{2}\left(C_{ijkl} + C_{ijlk}\right) + \frac{1}{2}\left(C_{ijkl} - C_{ijlk}\right) = C'_{ijkl} + C''_{ijkl}$, say

where $C'_{ijkl} = C'_{ijlk}$ and $C''_{ijkl} = -C''_{ijlk}$,

Which gives $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$ for symetry of T_{ij} and e_{kl}.

Hence C_{ijkl} has 36 components instead of 81. The matrix of the coefficients C_{ijkl} takes the form

$$\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2212} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{pmatrix}$$

Equation (1) is generatlized Hooke's law or stress-strain relation or constitutive equation of linearly elastic solid.

4.5 Strain Energy :

Physically when an elastic body is under the action of external surface forces, the body deform and external surface forces that act on the body do a certain amount of work. The work done in straining such an elastic body from the configuration of unstrained state to the present state by surface force is transformed completely into the potential energy stored in the body. This potential energy is due to deformation or strain only. It is called strain energy of the elastic body.

4.6 Existence of Strain Energy Function :

The principle of conservation of energy i.e., first law of thermodynamics gives

$$\rho\frac{de}{dt} = T_{ij}d_{ij} - q_{i,i} + \rho h \text{ .... (2)}$$

where e = internal energy per unit mass,

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

q_i = the flux heat by conduction per unit area per unit time,

h = rate per unit mass at which heat energy by radiation is produced from internal sources.

For linear elastic solid it is assumed that heat conduction is neglected and that heat energy is produced entirely by internal sources only. So, $q_i = 0$. Then from (2) we get,

$$\rho\frac{de}{dt} = T_{ij}d_{ij} + \rho h \text{ .... (3)}$$

Now, for small strains we have

$$d_{ij} = \dot{e}_{ij}$$

and if Q_1 be the quantity of heat per unit mass produced by internal sources at time t, then

$$h = \frac{dQ_1}{dt}.$$

Hence (3) reduces to,

$$T_{ij}\dot{e}_{ij} = \rho\left(\dot{e} - \dot{Q}_1\right) \text{..........} (4)$$

Now we introduce

$$U = \rho_0 e \text{ and } Q = \rho_0 Q_1 \text{.........} (5)$$

Here, U is the internal energy per unit volume of the unstrained state of the body and Q is the quantity of heat produced from internal sources per unit volume of the unstrained state. So, (5) takes the form

$$T_{ij}\dot{e}_{ij} = \frac{\rho}{\rho_0}\left(\dot{U} - \dot{Q}\right) \text{........} (6)$$

But $\frac{\rho}{\rho_0} = \frac{dV_0}{dV} = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \left|\frac{\partial x_i}{\partial X_j}\right| = \left|u_{i,j} + \delta_{ij}\right| = u_{i,i} + 1$. (for small displacements)

For small displacement gradients we have $\frac{\rho}{\rho_0} = 1$, i,e., $\rho = \rho_0$. Then (6) reduces into

$$T_{ij}\dot{e}_{ij} = \dot{U} - \dot{Q} \text{........} (7)$$

Hence first law of thermodynamics gives the above result.

Again, from the second law of thermodynamics, we have

$$\dot{Q} = \frac{\partial Q}{\partial t} = T\frac{\partial S}{\partial t} \text{..........} (8)$$

where S is the entropy and T is the temperature per unit volume.

If the change of state from one configuration to another takes place adiabatically, then the change takes place so rapidly that there is no time for the heat generated to the dissipated. This is referred as adiabatic process and consequently $\dot{Q} = 0$. Hence (7) gives

$$T_{ij}\dot{e}_{ij} = \dot{U}$$

or, $\quad T_{ij}de_{ij} = dU$ .... (9)

which shows that L.H.S. of (9) is an exact differential as R.H.S. is. So, there exist a function W such that

$$T_{ij}de_{ij} = dW \text{.........} (10)$$

where W is the internal energy of the body.

If the change of state takes place isothermally in which the change is so slow that heat generated has time enough to be dissipated so that temperature of the body remains constants and the body is in continued equilibrium of temperature with surrounding bodies, and consequently

$$\frac{\partial T}{\partial t} = \dot{T} = 0$$

Then (8) becomes

?
$$\dot{Q} = T\frac{\partial S}{\partial t} + S\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}(TS) \quad \text{.......... (11)}$$

Eliminating $\dot{Q}$ from (7) and (11), we get

$$T_{ij}\dot{e}_{ij} = \frac{\partial}{\partial t}[U - TS] = \frac{\partial F}{\partial t} = \dot{F}, \text{ say, } \quad \text{.......... (12)}$$

where $F = U - TS$, is the *Helmholtz's* free energy.

$$\therefore T_{ij}de_{ij} = dF \text{ (using (12)) } \quad \text{........ (13)}$$

which shows that L.H.S. is an exact differential. Therefore, there exist a function W such that

$$T_{ij}de_{ij} = dW \quad \text{.......... (14)}$$

Here, W represents the Helmholtz's free energy per unit volume of the elastic medium.

Now we introduce the following notations to avoid double sum:

$T_{11} = T_1, T_{22} = T_2, T_{33} = T_3$

$T_{23} = T_{32} = T_4,$

$T_{31} = T_{13} = T_5,$

$T_{12} = T_{21} = T_6,$

and $e_{11} = e_1, e_{22} = e_2, e_{33} = e_3,$

$2e_{23} = 2e_{32} = e_4,$

$2e_{31} = 2e_{13} = e_5,$

$2e_{12} = 2e_{21} = e_6.$

Then, we have,

$$T_{ij}de_{ij} = T_i de_i \; (i = 1, 2, \ldots, 6)$$

Hence, from (10) and (14), whether change of state is isothermal or adiabatic, there exist a function W such that

$$T_i de_i = dW \left(i = 1,2,...,6\right) \text{......... } (15)$$

Now $T_i de_i = T_{ij} de_{ij}$ re presents the work done per unit volume at a point by all surface forces, and therefore dW represents the work done per unit volume. If W is a function of independent variables $e_1, e_2, ..., e_6$, then

$$dW = \frac{\partial W}{\partial e_i} de_i \text{ .... } (16)$$

$$\therefore T_i de_i = \frac{\partial W}{\partial e_i} de_i$$

$$\Rightarrow T_i = \frac{\partial W}{\partial e_i} \left(i = 1,2,....,6\right) \text{ .... } (17)$$

Thus, for the both processes (isothermal and adiabatic) there exist a function W with the property $T_i = \frac{\partial W}{\partial e_i}$.

This function $W = W\left(e_1, e_2, ..., e_6\right)$ is called the *stress potential* or strain energy function for an unit volume of the elastic body, as it is the potential energy per unit volume stored up in the body by strain. Equation (16) gives stress components in terms of partial derivatives of strain energy w.r.t. corresponding strain components.

4.7 Istropic Elastic Solid :

A linearly elastic solid is known as to be isotropic *if it has the same elastic symmetry in all directions*. It menas the strain energy W must be invariant under all orthogonal transformations of co-ordinate axes. That is, in other words, W is independent of the orientation of co-ordinate axes and hence W must be expressed in terms of invariants of strain tensor.

Constitutive Equation :

The generalized Hooke's law for linearly elastic material is

$$T_{ij} = a_{ijkl} e_{kl} \left(i = 1,2,3; j = 1,2,3\right)$$

$$a_{ijkl} = \frac{1}{2}\left(a_{ijkl} + a_{ijlk}\right) + \frac{1}{2}\left(a_{ijkl} - a_{ijlk}\right)$$

$$= b_{ijkl} + c_{ijkl}, \text{ say}$$

where $b_{ijkl} = \frac{1}{2}\left(a_{ijkl} + a_{ijlk}\right) = b_{ijlk}$

and $c_{ijkl} = \frac{1}{2}\left(a_{ijkl} - a_{ijlk}\right) = -c_{ijlk}$

A tensor which has the same components in all rotated co-ordinate system is known as isotropic tensor. All rank-0 tensors (scalars) are isotropic.

$\therefore T_{ij} = b_{ijkl}e_{kl} + c_{ijkl}e_{kl}$

Also $c_{ijkl}e_{kl} = c_{ijlk}e_{lk}$ (interchanging dummy suffixes)

$= -c_{ijkl}e_{lk}$

$= -c_{ijkl}e_{kl}$ ($\because e_{kl}$ is symmetric)

$\Rightarrow c_{ijkl}e_{kl} = 0.$

Hence $T_{ij} = b_{ijkl}e_{kl}$

where $b_{ijkl} = b_{ijlk}$

Since T_{ij} and e_{kl} form second order tensor, and hence b_{ijkl} form a tensor of order four. For an isotropic elastic medium the elastic constants b_{ijkl} remains the same under all orthogonal transformation of co-ordinates axes. Thus for isotropic body b_{ijkl} must be an isotropic tensor of order four. Therefore, we can write

Project work → ?

$$b_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \quad \text{.......... (18)}$$

where λ, μ, γ are constants.

or, $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = \lambda\delta_{ij}\delta_{lk} + \mu\delta_{il}\delta_{jk} + \gamma\delta_{ik}\delta_{jl}$

$(\because b_{ijkl} = b_{ijlk})$

or, $(\mu-\gamma)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = 0$ for i ≠ j

which is true for all values of i, j, k, l.

Setting $i = 1, k = 1, j = 2, l = 2$ then above relation becomes

$$(\mu-\gamma)(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) = 0$$

or, $(\mu-\gamma) = 0$

i.e., $\mu = \gamma$

Therefore, (18) takes the form

$$b_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad \text{.......... (19)}$$

Now the generalized Hooke's law reduces to

$$T_{ij} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]e_{kl}$$

$$= \lambda\delta_{ij}e_{kk} + \mu(e_{il}\delta_{jl} + e_{kl}\delta_{jk})$$

$$= \lambda\delta_{ij}e_{kk} + \mu(e_{ij} + e_{ji})$$

$$\therefore T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \quad \text{............ (20)}$$

, where λ and μ are elastic constants known as Lame's constants;

where $\theta = e_{kk}$, is the first invariant.

Equations (20) are the *constitutive equations* or *stress-strain relation* for an isotropic linearly elastic body.

The number of elastic constants, in (20), are only two, namely λ and μ. Now the strain energy W is given by

?

$$W = \frac{1}{2}T_{ij}\, e_{ij} \left(\because dW = T_{ij} de_{ij}\right)$$

$$= \frac{1}{2}\left(\lambda\theta\delta_{ij} + 2\mu e_{ij}\right)e_{ij}$$

$$= \frac{1}{2}\lambda\theta^2 + \mu e_{ij} e_{ij}$$

$$= \frac{1}{2}\lambda\theta^2 + \mu\left(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{31}^2\right)$$

Thus strain energy W is a positive definite form in the strain e_{ij} taking positive values only.

Theorem : Principal directions of strain at each point of a linearly elastic isotropic body are coincident with the principal directions of stress.

Proof. : Let us consider the principal directions of strain at a point of the medium as co-ordinates axes. So, if e_{ij} and T_{ij} be the strain and stress tensors at that point, then we have

$$e_{12} = 0, e_{23} = 0, e_{31} = 0.$$

Also from the stress-strain relation for the linearly elastic isotropic body we have the constitutive equation as

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

$$\therefore T_{12} = \lambda\theta\delta_{12} + 2\mu e_{12} = 2\mu e_{12} = 0$$

$$\therefore T_{23} = \lambda\theta\delta_{23} + 2\mu e_{23} = 2\mu e_{23} = 0$$

and $T_{31} = \lambda\theta\delta_{31} + 2\mu e_{31} = 2\mu e_{31} = 0$

which implies that the co-ordinate axes must be along the principal directions of strain are coincides with the principal directions of stress for an isotropic body, that means for an isotropic body there is no distinction made between principal direction of strain and those of stress. Both are referred as principal directions.

Note 1. Steel, aluminium, glass are examples of isotropic material.

Note 2. A material elastically symmetry w.r.t. a plane with 13 elastic coefficients as $c_{11}, c_{12}, c_{13}, c_{16}, c_{22}, c_{23}, c_{26}, c_{36}, c_{44}, c_{45}, c_{66},$ is called *monotropic* material.

Note 3. A material having three mutually perpendicular planes of elastic symmetry is said to be *orthotropic.* W is an example of an orthotropic elastic material and in this case *W* is given by

$$W = \frac{1}{2}\left[c_{11}e_1^2 + c_{22}e_2^2 + c_{33}e_3^2 + c_{44}e_4^2 + c_{55}e_5^2 + c_{66}e_6^2 + 2c_{12}e_1e_2\right.$$

$$\left. + 2c_{13}e_1e_3 + 2c_{23}e_2e_3\right]$$

which contains only 9 non-zero elastic constants instead of 13 in case of monotropic material.

Note 4. A material with 21 elastic constants $c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{22}, c_{23}, c_{24}, c_{25}, c_{26}, c_{33}, c_{34}, c_{35}, c_{36}, c_{44}, c_{46}, c_{55}, c_{56}, c_{66}$ is called *anistropic* linerly elastic material. For an example of anisotropic body is provided crystal.

4.8 Basic Elastic Constants : for Isotropic Solid :

For the isotropic linear elastic material, the constitutive equation, is

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

where λ and μ are two elastic constants, known as *Lame's constants.*

$$\therefore T_{ii} = \lambda\theta\delta_{ii} + 2\mu e_{ii}$$

$$= 3\lambda\theta + 2\mu\theta \; (\because \theta = e_{ii})$$

$$= (3\lambda + 2\mu)\theta$$

or, $\Theta = (3\lambda + 2\mu)\theta$ where $T_{ii} = \Theta$

$$\therefore \theta = \frac{\Theta}{3\lambda + 2\mu} \quad \text{............ (21)}$$

Hence, $T_{ij} = \frac{\lambda\Theta}{3\lambda + 2\mu}\delta_{ij} + 2\mu e_{ij}$

or, $e_{ij} = \frac{T_{ij}}{2\mu} - \frac{\lambda\Theta\delta_{ij}}{2\mu(3\lambda + 2\mu)}$ .... (22)

which is the strain-stress relation and known as the *inversion of Hooke's law.*

Now, in particular, for *i*=1, *j*=1, we have

$$e_{11}=\frac{T_{11}}{2\mu}-\frac{\lambda\Theta}{2\mu(3\lambda+2\mu)}$$

$$=\frac{T_{11}}{2\mu}\left[1-\frac{\lambda}{3\lambda+2\mu}\right]-\frac{\lambda(T_{22}+T_{33})}{2\mu(3\lambda+2\mu)}$$

$$=\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\cdot T_{11}-\frac{\lambda}{2\mu(3\lambda+2\mu)}(T_{22}+T_{33})\text{ .... (23)}$$

Let us consider the situation $T_{11}=\text{Constant}=T,\ T_{22}=T_{33}=T_{23}=T_{31}=T_{12}=0$. This state of stress is possible in an elastic right circular cylinder the axis of which is parallel to x_1 axis and subjected to an uniform longitudinal axial tensile loading to both of its ends. Also, the above state of stress satisfies the equilibrium equations in absence of body forces at every point in the interior of the cylindrical elastic medium and also satisfies the stress-free boundary condition on its lateral surface.

Using (23) in (22), then we get,

$$\left.\begin{aligned} &e_{11}=\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}T,\ e_{22}=e_{33}=-\frac{\lambda}{2\mu(3\lambda+2\mu)}\cdot T,\\ &e_{23}=0,\ e_{31}=0,\ e_{12}=0.\end{aligned}\right\}\text{ .... (24)}$$

Now, the ratio of the tensile stress T_{11} to the longitudinal extension e_{11}, i.e.,

$$\frac{T_{11}}{e_{11}}=\frac{T}{e_{11}}=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}=\text{Constant.}$$

This ratio is called *Young's modules* or *modulus of elasticity*, and is denoted by E. Hence, by definition we have

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\text{ .... (25)}$$

Also, we consider another ratio of lateral contraction to the longitudinal extension, i.e.,

$$-\frac{e_{22}}{e_{11}}=\frac{\lambda}{2(\lambda+\mu)}=\text{Constant.}$$

This ratio is called *Poisson's ratio*, and is denoted by σ.

Hence

$$\sigma=\frac{\lambda}{2(\lambda+\mu)}\text{ .... (26)}$$

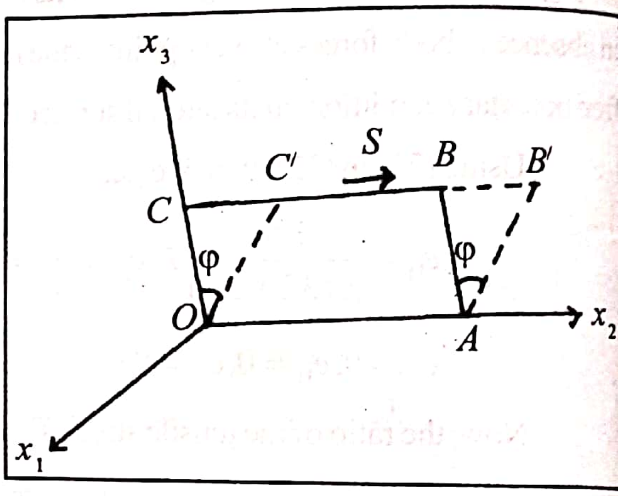
Now, from (25) and (26) we get

$$\left.\begin{aligned} \lambda &= \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \\ \text{and} \quad \mu &= \frac{E}{2(1+\sigma)} \end{aligned}\right\} \quad \text{..........} (27)$$

Next we consider the state of stress

$$T_{11} = T_{22} = T_{33} = T_{12} = T_{31} = 0,\ T_{23} = S = \text{Constant} \text{........} (28)$$

This state of stress is possible in a deformed long rectangular parallelopiped of square cross-section $OABC$ which is sheared in the plane containing OA and OC by a shearing stress of magnitude S acting per unit area on the side CB. Now the stress S would tend to slide the planes of the material originally perpendicular to OC, the x_3-axis, in a direction parallel to OA, the x_2-axis so that the right angle between OA and OC will diminished by an angle φ. This state of stress satisfies the equations of equilibrium in the absence of body force at every point in the interior and the boundary condition on the surface.



Now from (22), using (28), we get

$$e_{23} = \frac{T_{23}}{2\mu} = \frac{S}{2\mu}, e_{11} = e_{22} = e_{33} = e_{31} = e_{12} = 0 \text{..........} (29)$$

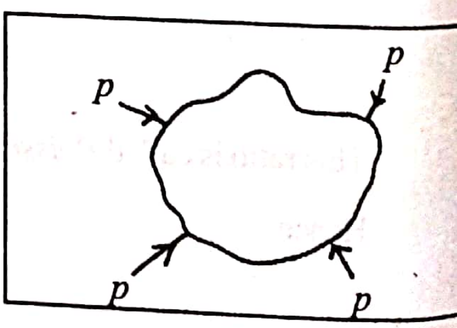
Also, from the definition

$$2e_{23} = \varphi = \text{Change in angle}$$

$$\therefore \frac{S}{\varphi} = \frac{S}{2e_{23}} = \mu \text{..........} (30)$$

This ratio is known as *shear modulus* or *modulus of rigidity*, and it is identical with Lame's constant μ.

Finally consider an elastic body of arbitrary shape subjected to a hydrostatic stress p distributed over its surface. Also the hydrostatic stress diminishes the volume of the body. This state of stress possible in such a



deformed body is given by

$$T_{11}=T_{22}=T_{33}=-p=\text{constant.}$$

$$T_{23}=T_{31}=T_{12}=0$$

The state of stress satisfies equilibrium equation in the interior of body. If $T_i^{(n)}$ be stress vector acting on the surface with normal n_i, then

$$T_i^{(n)}=-pn_i\left(i=1,2,3\right)$$

and hence

$$T_{ij}n_i=-pn_i \text{ .... (32)}$$

at each and every point on the surface. It is obvious that state of stress in (31) also satisfies the boundary condition on the surface.

Using (31) into (22), we get

$$\left.\begin{aligned} e_{11}=e_{22}=e_{33}&=-\frac{p}{2\mu}+\frac{3\lambda p}{2\mu\left(3\mu+2\mu\right)}=-\frac{p}{3\lambda+2\mu}=\text{constant.}\\ e_{12}=e_{23}=e_{31}&=0.\end{aligned}\right\}\text{ .... (33)}$$

Now, if θ be the cubical dilatation, then the decrease in volume per unit volume is $\left(-\theta\right)$.

So,

$$-\theta=-e_{ii}=-\left(e_{11}+e_{22}+e_{33}\right)$$

$$=\frac{3p}{3\lambda+2\mu}.$$

Thus the ratio of the hydrostatic stress to the decrease in volume per unit volume, i.e.,

$$\frac{p}{-\theta}=\frac{3\lambda+2\mu}{3}=\lambda+\frac{2\mu}{3}=\text{constant.}$$

This constant is known as *bulk modulus* or *modulus of compression*, and is denoted by K. So,

$$K=\lambda+\frac{2\mu}{3} \text{ .... (34)}$$

Also K can be express in terms of E and σ as

$$K=\frac{E}{3\left(1-2\sigma\right)} \text{ .... (35)}$$

If $K>0$, $E>0$, then we must have from (35)

$$0<\sigma<\frac{1}{2}$$

Again, since $0<\sigma<\frac{1}{2}$ and $E>0$, then we have $\lambda>0, \mu>0$.

For most general materials σ does not deviate from $\frac{1}{3}$ and if the material is incompressible

$$\theta=0, K\to\infty, \sigma=\frac{1}{2}, \mu=\frac{E}{3} \quad\text{.........}\ (36)$$

For some solids and rocks $\lambda=\mu$ and in this case

$$K=\frac{5\mu}{3}, E=\frac{5\mu}{2}, \gamma=\frac{1}{4}.$$

Now, from (25) and (26), we have

$$\frac{1+\sigma}{E}=\frac{1}{2\mu}$$

and $$\frac{\sigma}{E}=\frac{\lambda}{2\mu(3\lambda+2\mu)}.$$

Hence the stress-strain relation becomes

$$e_{ij}=\frac{1+\sigma}{E}T_{ij}-\frac{\sigma\,\Theta\,\delta_{ij}}{E} \quad\text{......................}\ (37)$$

and cubical dilatation θ is

$$\theta=e_{ii}=\frac{1+\sigma}{E}T_{ii}-\frac{3\sigma}{E}\Theta$$

$$=\frac{1+\sigma}{E}\cdot\Theta-\frac{3\sigma}{E}\cdot\Theta$$

$$=\frac{1-2\sigma}{E}\cdot\Theta$$

or, $$\theta=\frac{\Theta}{3K} \ \text{(using 35)}$$

$$\therefore \frac{\Theta}{3}=K\theta \quad\text{..........}\ (38)$$