Study Material

Dept. of Applied Mathematics with Oceanology and Computer Eng.

Paper No. - MTM 205

Paper No. - Continuum Mechanics

Semester - 2

Topic of Lecture;

Energy Equation of a Perfect Fluid

Teacher: Prof. Shyamal Kr Mondal

Lecture No. 01.

Corollary - 1. In a closed circuit C of fluid particles moving under the same conditions as before, $\int_S \vec{n}\cdot(\vec{\nabla}\times\vec{V})dS = \int_S \vec{n}\cdot\vec{W}dS$ is constant, where S is an open surface with C as rim.

Proof. The circulation along the closed path C, for all time t, is $\Gamma = \oint_C \vec{V}\cdot d\vec{r} = \int_S \vec{n}\cdot(\vec{\nabla}\times\vec{V})dS$, (by Stoke's theorem) which implies that $\int_S \vec{n}\cdot(\vec{\nabla}\times\vec{V})dS = \int_S \vec{n}\cdot\vec{W}dS =$ constant as Γ is constant (under the conditions of Kelvin's theorem).

Corollary - 2. Under the above conditions, if any portion of the moving fluid once becomes irrotational, then it remains so for all subsequent times.

Proof. Let us suppose that at some instant the fluid on S becomes irrotational. Then $\vec{V} = -\vec{\nabla}\varphi$ and hence the vorticity vector $\vec{W} = \vec{\nabla}\times\vec{V} = -\vec{\nabla}(\vec{\nabla}\varphi) = \vec{0}$ at all points of S and the last result shows that

$$\Gamma = \oint_C \vec{V}\cdot d\vec{r} = \int_S \vec{n}\cdot\vec{W}dS = 0 \text{ for } \vec{W} = \vec{0}$$

at the given instant and hence at all subsequent times.

Thus at any later stage,

$$\int_S \vec{n}\cdot\vec{W}dS = 0$$

If we now take S to be a non-zero infinitesimal element, say dS, then to the first order $\vec{n}\cdot\vec{W}dS = 0$ showing that in general $\vec{W} = \vec{0}$ at any point of dS, i.e., that the motion stays irrotational.

Corollary - 3. With the above conditions, the vortex lines moves along with the fluid.

Proof. For a surface S moving with the fluid we have seen that $\int_S \vec{n}\cdot\vec{W}dS =$ constant. This integral represents total strength of vortex tubes passing through S. Which shows that the vortex tubes move with the fluid. By taking S vanishingly small, we infer that the vortex lines move with the fluid.

6.2 Energy Equation :

The time rate of change of total energy (i.e. sum of kinetic, potential, intrinsic energies) of any portion of the fluid is equal to the rate of work done by external pressure across its boundary provided the external forces are conservative, i.e.

$$\frac{d}{dt}(T+W+I)=\iint p(\vec{n}\cdot\vec{v})\,dS\,,$$

where T, W, I are Kinetic energy, Potential energy, Intrinsic energy respectively.

Proof. Let us consider an arbitrary closed surface S moving with a non-viscous fluid such that it encloses a volume V. Let $\vec{n}$ be the unit inward drawn normal vector on an element dS. Let the force be conservative so that $\vec{F}=-\vec{\nabla}\Omega$ where Ω is time-independent scalar potential function.

So, $\dfrac{\partial\Omega}{\partial t}=0$ and hence

$$\frac{d\Omega}{dt}=\frac{\partial\Omega}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\Omega=\left(\vec{V}\cdot\vec{\nabla}\right)\Omega \quad ...(i)$$

Let T, W and I denotes the kinetic energy, potential energy and intrinsic energy respectively. Since Ω is force potential per unit mass and hence

$$W=\int\Omega\,dm=\int_V\Omega\rho\,dV$$

$$T=\int_V\frac{1}{2}\rho\left|\vec{V}\right|^2dV=\int_V\frac{1}{2}\left|\vec{V}\right|^2\rho\,dV$$

Since elementary mass remains invariant through the motion and so $\dfrac{d}{dt}(\rho dV)=0$.

$$\text{Now, }\frac{dT}{dt}=\frac{1}{2}\int_V\frac{d\left|\vec{V}\right|^2}{dt}\cdot\rho dV+\frac{1}{2}\int_V\left|\vec{V}\right|^2\frac{d(\rho dV)}{dt}$$

$$=\frac{1}{2}\int_V\vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV+0\left(\because\left|\vec{V}\right|^2=\vec{V}\cdot\vec{V}\ \text{ and }\ \frac{d}{dt}(\rho dV)=0\right)$$

$$\text{and }\frac{dW}{dt}=\int_V\frac{d\Omega}{dt}\rho dV+\int_V\Omega\frac{d(\rho dV)}{dt}=\int_V\frac{d\Omega}{dt}\rho dV\,.$$

Intrinsic energy E per unit mass of the fluid is defined as the work done by the unit mass of the fluid against external pressure p under the supposed relation between pressure and density ρ from its actual state to some standard state in which pressure and density are p_0 and ρ_0 respectively. Then

$$I=\int_V E\rho\,dV$$

where $E = \int_V^{V_0} p dV \quad = \int_\rho^{\rho_0} p d\left(\frac{1}{\rho}\right)\left(\because \rho = \frac{1}{V}\right)$

$$= -\int_\rho^{\rho_0} p \frac{d\rho}{\rho^2} = \int_{\rho_0}^{\rho} \frac{p}{\rho^2} d\rho$$

$$\Rightarrow \frac{dE}{d\rho} = \frac{p}{\rho^2} \quad \text{... (ii)}$$

$$\therefore \frac{dI}{dt} = \int_V \left[\frac{dE}{dt}\rho dV + E\frac{d}{dt}(\rho dV)\right] = \int_V \frac{dE}{dt}\rho dV$$

$$= \int_V \frac{dE}{d\rho}\cdot\frac{d\rho}{dt}\rho dV = \int_V \frac{p}{\rho^2}\frac{d\rho}{dt}\rho dV \quad \text{[using (ii)]}$$

$$= \int_V \frac{p}{\rho}\frac{d\rho}{dt} dV = \int_V \frac{p}{\rho}\left(-\rho\vec{\nabla}\cdot\vec{V}\right)dV \left(\because \frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{V} = 0, \text{ equation of continuity}\right)$$

i.e., $\frac{dI}{dt} = -\int_V p\left(\vec{\nabla}\cdot\vec{V}\right)dV$

Hence, finally we have

$$\frac{dT}{dt} = \int_V \vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV \quad \text{... (iii)}$$

$$\frac{dW}{dt} = \int_V \frac{d\Omega}{dt}\rho dV, \quad \text{... (iv)}$$

and $\frac{dI}{dt} = -\int_V p\left(\vec{\nabla}\cdot\vec{V}\right)dV \quad \text{... (v)}$

Again the Euler's equation of motion for conservative field is

$$\frac{d\vec{V}}{dt} = -\vec{\nabla}\Omega - \frac{1}{\rho}\vec{\nabla}p$$

So, $\vec{V}\cdot\frac{d\vec{V}}{dt} = \vec{V}\cdot\left[-\vec{\nabla}\Omega - \frac{1}{\rho}\vec{\nabla}p\right] = -\left(\vec{V}\cdot\vec{\nabla}\right)\Omega - \frac{1}{\rho}\vec{V}\cdot\vec{\nabla}p$

or, $\vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV = -\left[\left(\vec{V}\cdot\vec{\nabla}\right)\Omega\right]\rho dV - \left(\vec{V}\cdot\vec{\nabla}p\right)dV$

Integrating over V,

$$\int_V \vec{V}\cdot\frac{d\vec{V}}{dt}\rho dV = -\int_V (\vec{V}\cdot\vec{\nabla})\Omega\rho dV - \int_V (\vec{V}\cdot\vec{\nabla}p)dV$$

or, $$\frac{dT}{dt} = -\int_V \frac{d\Omega}{dt}\rho dV - \int_V (\vec{V}\cdot\vec{\nabla}p)dV \text{ [using (i) and (iii)]}$$

or, $$\frac{dT}{dt} = -\frac{dW}{dt} - \int_V (\vec{V}\cdot\vec{\nabla}p)dV \text{ [using (iv)]}$$

or, $$\frac{dT}{dt} + \frac{dW}{dt} = -\int_V (\vec{V}\cdot\vec{\nabla}p)dV \quad \text{... (vi)}$$

But we know that the vector identity

$$\vec{\nabla}\cdot(p\vec{V}) = p\vec{\nabla}\cdot\vec{V} + (\vec{V}\cdot\vec{\nabla})p$$

$$\therefore \int_V \vec{\nabla}\cdot(p\vec{V})dV = \int_V p(\vec{\nabla}\cdot\vec{V})dV + \int_V (\vec{V}\cdot\vec{\nabla})pdV$$

or, $$\int_S -\vec{n}\cdot(p\vec{V})dS = -\frac{dI}{dt} + \int_V (\vec{V}\cdot\vec{\nabla})pdV$$ (using Gauss's div. theorem and (v), $\vec{n}$ is inward normal)

or, $$-\int_S \vec{n}\cdot(p\vec{V})dS + \frac{dI}{dt} = \int_V (\vec{V}\cdot\vec{\nabla})pdV \quad \text{... (vii)}$$

Using (vii) in (vi) we get

$$\frac{dT}{dt} + \frac{dW}{dt} = -\left[-\int_S \vec{n}\cdot(p\vec{V})dS + \frac{dI}{dt}\right]$$

or, $$\frac{dT}{dt} + \frac{dW}{dt} + \frac{dI}{dt} = \int_S \vec{n}\cdot(p\vec{V})dS \quad \text{... (viii)}$$

which is known as equation of energy of a perfect fluid.

Corollary 1: Energy equation for incompressible fluid.

Proof. In case of incompressible fluid we have

$$\vec{\nabla}\cdot\vec{V} = 0$$

So, from (v), we get

$$\frac{dI}{dt} = 0$$

Hence from (viii) we have

$$\frac{dT}{dt}+\frac{dW}{dt}=\int_s \vec{n}\cdot\left(p\vec{V}\right)dS \quad \text{.......... (ix)}$$

which is the equation of energy for incompressible fluid.

Corollary 2: Energy equation for incompressible fluid within the fixed boundary

Proof. If the fluid is bounded on all sides by fixed boundary so that the fluid moves only tangentially over the surface, i.e., there is no normal flow across the boundary, then $\vec{n}\cdot\vec{V}=0$ at even point on *S*.

Now from (ix) i.e., equation of energy for incompressible fluid, we get

$$\frac{dT}{dt}+\frac{dW}{dt}=\int_s \vec{n}\cdot\left(p\vec{V}\right)dS=0$$

i.e. $$\frac{d\left(T+W\right)}{dt}=0$$

or, $T+W=\text{contsnt}$

Hence, if incompressible fluid is contained within a fixed boundary, the sum of its kinetic and potential energies remain unchanged with the passage of time.

6.13 Kelvin's Minimum Energy Theorem:

The irrotational motion of an incompressible fluid occupying a simply connected region has less kinetic energy than any other motion of the fluid for which fluid has on the boundary same normal velocity as irrotational motion.

Proof. Let us consider an incompressible fluid occupying a simply-connected region *V* bounded by the closed surface *S*. Let ρ be density of the fluid, *T* the kinetic energy of the fluid moving irrotationally in which *u*, *v*, *w* are components of fluid velocity. Then

$$T=\int_V \frac{1}{2}\left|\vec{V}\right|^2 \rho dV=\frac{1}{2}\rho\int_V\left(u^2+v^2+w^2\right)dV \quad \text{...... (i)}$$

Since the fluid motion is irrotational with velocity potential φ, therefore,

$$\vec{V}=-\vec{\nabla}\varphi$$

i.e. $$u=-\frac{\partial\varphi}{\partial x},\quad v=-\frac{\partial\varphi}{\partial y},\quad w=-\frac{\partial\varphi}{\partial z} \quad \text{...... (ii)}$$

at everywhere.