Study Material

Dept. of Applied Mathematics

With Oceanology and Computer Prog.

Paper NO. - MTM 205

Paper NO. - Continuum Mechanics

Semester - 2

Topic of Lecturer: Permanence of Irrotation Motion and Kelvin's Minimum Energy Theorem

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Lecturer No. 02

7.4 Permanence of Irrotational Motion

Theorem: If any portion of the moving fluid once becomes irrotational, then it will remain so for all subsequent times provided that the external body forces are conservative and pressure is a function of density alone.

Proof: Consider a simply-connected region in the fluid and draw a closed curve C in that region. We can always draw a surface S with curve C as rim lying entirely in the fluid (Fig. 45).

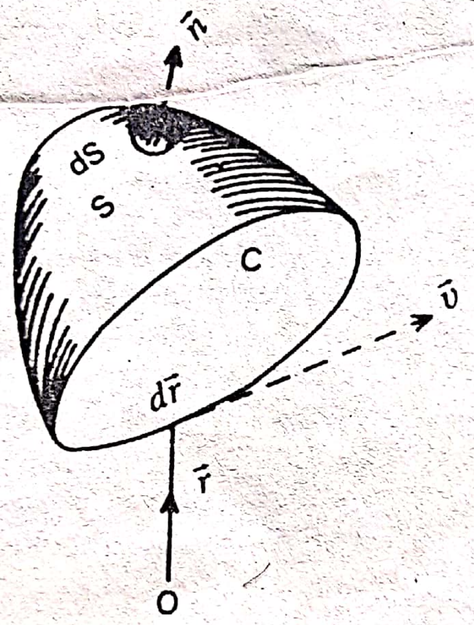

Fig. 45

The circulation $\Gamma = \oint_C \bar{v} \cdot d\bar{r}$, where $d\bar{r}$ is the directed element at any point of C, and $\bar{v}$ the fluid velocity.

Now by Stoke's theorem

$$\Gamma = \oint_C \bar{v} \cdot d\bar{r} = \int_S \bar{n} \cdot \text{rot}\, \bar{v}\, dS \qquad (1)$$

where dS is the element of the surface S and $\bar{n}$ is unit normal to dS.

Suppose that at initial instant of time fluid motion is irrotational. So that

$$\text{rot}\, \bar{v} = 0 \text{ at all points of } S \qquad (2)$$

It follows from (1)

$$\Gamma = 0 \text{ at the initial instant of time} \qquad (3)$$

By Kelvine's theorem on the constancy of circulation of a perfect fluid, the circulation round a closed curve remains constant for all time provided the external forces are conservative and pressure is a function of density alone.

Since initially circulation $\Gamma = 0$, it will be equal to zero for all subsequent time. It follows from (1)

$$\int_S \vec{n} \cdot \text{rot}\, \vec{v}\, dS = 0 \text{ at any subsequent time} \tag{4}$$

This can be true if it turns out that $\vec{n} \cdot \text{rot}\, \vec{v} = 0$ at every point of S and for any direction $\vec{n}$ at any subsequent time.

In other words, at any subsequent time we must have rot $\vec{v} = 0$ at every point of S. Thus motion stays irrotational.

7.5 Kelvin's Minimum Energy Theorem

Theorem: The irrotational motion of an incompressible fluid occupying a simply connected region has less kinetic energy than any other motion of the fluid for which fluid has on the boundary same normal velocity as irrotational motion.

Proof: Consider an incompressible fluid occupying a simply-connected region V bounded by the closed surface S. Let ρ be the density of the fluid, and $\mathscr{K}$ the kinetic energy of the fluid moving irrotationally in which v_1, v_2, v_3 are components of fluid velocity. Then

$$\mathscr{K} = \frac{1}{2}\rho \int_V (v_1^2 + v_2^2 + v_3^2)\, dV \tag{1}$$

Since fluid motion is irrotational with velocity potential φ,

$$v_i = \frac{-\partial\varphi}{\partial x_i} \text{ everywhere} \tag{2}$$

The velocity components v_i must satisfy the equation of continuity of incompressible fluid

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \tag{3}$$

at every interior point

Let $\mathscr{K}'$ be the kinetic energy of any other possible state of motion of the fluid in which velocity components are v_1', v_2', v_3'. Then

$$\therefore \quad \mathscr{K}' = \frac{1}{2}\rho \int_V (v_1'^2 + v_2'^2 + v_3'^2)\, dV \tag{4}$$

where velocity components satisfy the equation of continuity

$$\frac{\partial v_1'}{\partial x_1} + \frac{\partial v_2'}{\partial x_2} + \frac{\partial v_3'}{\partial x_3} = 0 \tag{5}$$

at every interior point

Let n_1, n_2, n_3 be direction cosines of the normal at any point on the boundary surface S. Since two motions have the same normal velocity on S,

$$n_1 v_1 + n_2 v_2 + n_3 v_3 = n_1 v_1' + n_2 v_2' + n_3 v_3' \tag{6}$$

Now $\mathscr{K}' - \mathscr{K} = \frac{\rho}{2}\int_V [(v_1'^2 - v_1^2) + (v_2'^2 - v_2^2) + (v_3'^2 - v_3^2)]\, dV$

$$= \frac{\rho}{2}\int_V [(v_1' - v_1)^2 + 2v_1(v_1' - v_1) + (v_2' - v_2)^2$$

$$+ 2v_2(v_2' - v_2) + (v_3' - v_3)^2 + 2v_3(v_3' - v_3)]\, dV$$

$$= \frac{\rho}{2}\int_V [(v_1' - v_1)^2 + (v_2' - v_2)^2 + (v_3' - v_3)^2]\, dV$$

$$+ \rho \int [v_1(v_1' - v_1) + v_2(v_2' - v_2) + v_3(v_3' - v_3)]\, dV$$

$$= I_1 + I_2 \text{ say} \tag{7}$$

where $I_2 = \rho \int_V [v_1(v_1' - v_1) + v_2(v_2' - v_2) + v_3(v_3' - v_3)]\, dV$ and I_1 is a

positive quantity given by $\frac{\rho}{2}\int_V [(v_1' - v_1)^2 + (v_2' - v_2)^2 + (v_3' - v_3)^2]\, dV$

Now $I_2 = -\rho \int_V \left[\frac{\partial \varphi}{\partial x_1}(v_1' - v_1) + \frac{\partial \varphi}{\partial x_2}(v_2' - v_2) + \frac{\partial \varphi}{\partial x_3}(v_3' - v_3)\right] dV$

(using (2))

$$= -\rho \int_V \left[\frac{\partial}{\partial x_1}\{\varphi(v_1' - v_1)\} + \frac{\partial}{\partial x_2}\{\varphi(v_2' - v_2)\} + \frac{\partial}{\partial x_3}\{\varphi(v_3' - v_3)\}\right] dV$$

$$+ \rho \int_V \varphi \left[\frac{\partial}{\partial x_1}(v_1' - v_1) + \frac{\partial}{\partial x_2}(v_2' - v_2) + \frac{\partial}{\partial x_3}(v_3' - v_3)\right] dV \tag{8}$$

It follows from (3) and (5)

$$\frac{\partial}{\partial x_1}(v_1' - v_1) + \frac{\partial}{\partial x_2}(v_2' - v_2) + \frac{\partial}{\partial x_3}(v_3' - v_3) = 0 \qquad (9)$$

Substituting (9) into (8)

$$I_2 = -\rho \int_V \left[\frac{\partial}{\partial x_1}\{\varphi (v_1' - v_1)\} + \frac{\partial}{\partial x_2}\{\varphi (v_2' - v_2)\} + \frac{\partial}{\partial x_3}\{\varphi (v_3' - v_3)\} \right] dV$$

$$= -\rho \int_S [n_1 \varphi (v_1' - v_1) + n_2 \varphi (v_2' - v_2) + n_3 \varphi (v_3' - v_3)]\, dS$$

using Gauss's divergence theorem.

$$\therefore \quad I_2 = -\rho \int_S \varphi [n_1(v_1' - v_1) + n_2(v_2' - v_2) + n_3(v_3' - v_3)]\, dS$$

$= 0$ using (6)

It follows from (7) $\mathcal{K}' - \mathcal{K} = I_1$ = a positive quantity

$$\therefore \quad \mathcal{K}' - \mathcal{K} > 0$$

or

$$\mathcal{K}' > \mathcal{K} \text{ or } \mathcal{K} < \mathcal{K}'$$

7.6 Extremum Value of the Velocity Potential

Theorem: The maximum or minimum value of the velocity potential for an irrotational flow of an incompressible fluid can occur only on the boundary of the fluid.

Let S be any arbitrary closed surface enclosing a volume V lying entirely within the fluid. Let φ denote the velocity potential of irrotational motion of an incompressible fluid. Now

$$\int_S \frac{\partial \varphi}{\partial n}\, dS = \int_V \nabla^2 \phi \, dV \qquad (1)$$

by Gauss's Divergence theorem.

Since φ satisfies Laplace's equation

$$\nabla^2 \varphi = 0 \qquad (2)$$

which is the equation of continuity. Equation (1) reduces to

$$\int_S \frac{\partial \varphi}{\partial n}\, dS = 0 \qquad (3)$$

Let P be any interior point in the fluid. Let us assume that φ is maximum

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at P. Let us surround this point P by sufficiently small closed surface S so that value of φ at P would be greater than the value of φ at all points of S.

Therefore at every point of S, $\frac{\partial \varphi}{\partial n} > 0$. Hence

$$\int_S \frac{\partial \varphi}{\partial n} dS > 0 \tag{4}$$

Result (4) contradicts (3). Therefore, φ cannot have a maximum value at a point within the fluid. Similarly, there cannot be a point within the fluid at which φ has a minimum value.

7.7 Maximum Value of the Speed

Theorem: In irrotational motion of an incompressible fluid, the maximum value of the speed occur on the boundary of the fluid.

Let P be any point within the fluid. We choose the coordinate axes such that axis of x_1 is in the direction of the motion of the fluid at P. If φ is the velocity potential and v_p is the speed at $P(x_1, x_2, x_3)$ then

$$v_P^2 = \left(\frac{\partial \varphi}{\partial x_1}\right)_P^2 \tag{1}$$

and
$$\nabla^2 \varphi = 0 \tag{2}$$

Let
$$\varphi_{,1} = \frac{\partial \varphi}{\partial x_1}$$

$$\nabla^2 \varphi_{,1} = \nabla^2\left(\frac{\partial \varphi}{\partial x_1}\right) = \frac{\partial}{\partial x_1}(\nabla^2 \varphi) = 0 \tag{3}$$

using (2).

Since $\varphi_{,1}$ satisfies Laplace's equation, it can be regarded as the possible velocity potential of some fluid motion. Therefore velocity potential φ can not have a maximum value at P. Consequently there must be another point Q in the neighbourhood of P for which

$$(\varphi_{,1})_Q > (\varphi_{,1})_P$$

or
$$\left(\frac{\partial \varphi}{\partial x_1}\right)_Q^2 > \left(\frac{\partial \varphi}{\partial x_1}\right)_P^2$$

or $$\left(\frac{\partial\varphi}{\partial x_1}\right)_Q^2+\left(\frac{\partial\varphi}{\partial x_2}\right)_Q^2+\left(\frac{\partial\varphi}{\partial x_3}\right)_Q^2>\left(\frac{\partial\varphi}{\partial x_1}\right)_P^2=v_P^2 \tag{4}$$

If v_Q is the speed at Q, then it follows from (4) $v_Q^2 > v_p^2$. Hence v_p cannot be a maximum. The maximum value of fluid speed, if any must occur on the boundary.

7.8 Kinetic Energy of the Incompressible Fluid Moving Irrotationally

In order to prove the uniqueness of the solution of an irrotational motion of a fluid with a given set of boundary condition we have to determine on expression for kinetic energy of the fluid in irrotational motion.

Consider an incompressible fluid of density ρ occupying some simply-connected region V bounded by closed surface S. If $\bar{v}$ be the velocity of the fluid, then kinetic energy $\mathscr{K}$ of the fluid is given by

$$\mathscr{K}=\frac{1}{2}\rho\int_V \bar{v}\cdot\bar{v}\,dV \tag{1}$$

Since motion is irrotational, there exists a velocity potential φ such that

$$\bar{v}=-\vec{\nabla}\varphi \text{ when } \nabla^2\varphi=0 \tag{2}$$

$$\therefore \qquad \mathscr{K}=\frac{1}{2}\rho\int_V [(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi)]\,dV \tag{3}$$

Now $$\vec{\nabla}\cdot(\varphi\vec{\nabla}\varphi)=(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi)+\varphi\nabla^2\varphi \tag{4}$$

Substituting (4) in (3)

$$\mathscr{K}=\frac{1}{2}\rho\int_V [\vec{\nabla}\cdot(\varphi\vec{\nabla}\varphi)-\varphi\nabla^2\varphi]\,dV$$

$$=\frac{1}{2}\rho\int_V \vec{\nabla}\cdot(\varphi\vec{\nabla}\varphi)\,dV, \text{ using (2)}$$

$$=-\frac{1}{2}\rho\int_S \vec{n}\cdot(\varphi\vec{\nabla}\varphi)\,dS \text{ by Gauss's theorem}$$

where $\vec{n}$ is unit normal to surface element dS drawn inward.

$$\therefore \qquad \mathscr{K}=-\frac{1}{2}\rho\int_S \phi(\vec{n}\cdot\vec{\nabla}\varphi)\,dS=-\frac{1}{2}\rho\int_S \varphi\frac{\partial\varphi}{\partial n}\,dS$$

7.9 Uniqueness Theorem

Theorem. There cannot be two different forms of irrotational motion for a given confined mass of liquid whose boundaries have prescribed velocities.

Suppose that two such different irrotational motions are possible. Let φ_1 and φ_2 denote two different velocity potentials corresponding to two different irrotational motions. Each of them must satisfy the equation of continuity

$$\nabla^2\varphi_1 = 0,\ \nabla^2\varphi_2 = 0 \text{ at every interior point} \tag{1}$$

Since in two motions, boundaries have the same prescribed normal velocities

$$\frac{\partial\varphi_1}{\partial n} = \frac{\partial\varphi_2}{\partial n} \text{ at every point on boundary} \tag{2}$$

Let us take

$$\varphi = \varphi_1 - \varphi_2 \tag{3}$$

$$\therefore \quad \nabla^2\varphi = \nabla^2\varphi_1 - \nabla^2\varphi_2 = 0 \text{ using (1) at every interior point.} \tag{4}$$

Therefore φ will represent a possible velocity potential of an irrotational motion in which kinetic energy is

$$\mathscr{K} = -\frac{1}{2}\rho\int_S \varphi\frac{\partial\varphi}{\partial n}\,dS \tag{5}$$

Now

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi_1}{\partial n} - \frac{\partial\varphi_2}{\partial n} = 0 \text{ at every point on boundary} \tag{6}$$

Using (6) in (5), $\mathscr{K} = 0$. Hence

$$\frac{1}{2}\rho\int_V v^2\,dV = 0$$

or

$$\int_V\left[\left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2 + \left(\frac{\partial\varphi}{\partial x_3}\right)^2\right]dV = 0$$

$$\frac{\partial\varphi}{\partial x_1} = 0,\ \frac{\partial\varphi}{\partial x_2} = 0,\ \frac{\partial\varphi}{\partial x_3} = 0 \text{ everywhere}$$

$$\varphi = \text{Constant everywhere}$$

$$\varphi_1 = \varphi_2 + \text{Constant}$$

indicating that φ_1 and φ_2 can differ only by a constant. Therefore the velocity distribution given by φ_1 and φ_2 are identical and two motions are identical.