The Algebraic Method

Let us consider the same example and illustrate the algebraic method.

Maximize $Z = 6x_1 + 5x_2$

Subject to $x_1 + x_2 \leq 5$

$3x_1 + 2x_2 \leq 12$

Assuming that we know to solve linear equations, we convert the inequalities into equations by adding slack variables x_3 and x_4 respectively.

These two slack variables do not contribute to the objective function. The linear programming problem becomes

Maximize $Z = 6x_1 + 5x_2 + 0 \cdot x_3 + 0 \cdot x_4$

Subject to $x_1 + x_2 + x_3 = 5$

$3x_1 + 2x_2 + x_4 = 12$

With the addition of the slack variables, we now have four variables and two equations. With two equations, we can solve only for two variables.

We have to fix any two variables to some arbitary value and can solve for remaining two variables.

The two variables that we fix arbitarily values can be chosen in ${}^4C_2 = 6$ ways.

In each of these six combinations, we can actually fix the variables to any arbitary value resulting in infinite number of solutions.

- However, we consider fixing the arbitary value to zero and hence consider only six distinct possible solutions.

The variable that we fix to zero are called non-basic variables and the variables that we solve are called basic variables.

- These solutions obtained by fixing the non basic variables to zero are called basic solutions.

The six basic solutions are:

1. Variables x_1 and x_2 are non-basic and set to zero. Substituting we get $x_3 = 5$, $x_4 = 12$ and the value of the objective function $Z = 0$
2. Variables x_1 and x_3 are non-basic and set to zero. Substituting, we solve for $x_2 = 5$ and $2x_2 + x_4 = 12$ and get $x_2 = 5$, $x_4 = 2$ and value of objective function $Z = 25$.
3. Variables x_1 and x_4 are non basic and set to zero. Substituting, we solve $x_2 + x_3 = 5$ and $2x_2 = 12$ and get $x_2 = 6$, $x_3 = -1$
4. Variables x_2 and x_3 are non-basic and set to zero. Substituting, we solve for $x_1 = 5$ and $3x_1 + x_4 = 12$ and get $x_1 = 5$, $x_4 = -3$.
5. Variables x_2 and x_4 are non basic and set to zero. Substituting, we solve $x_1 + x_3 = 5$ and $3x_1 = 12$ and get $x_1 = 4$, $x_3 = 1$ and value of objective function $Z = 24$.
6. Variables x_3 and x_4 are non basic and set to zero. Substituting, we solve for $x_1 + x_2 = 5$ and $3x_1 + 2x_2 = 12$ and get $x_1 = 2$, $x_2 = 3$ and value of objective function $Z = 27$

Among these ~~ba~~ six basic solutions, we observe that four are feasible

- Those basic solutions that are feasible (satisfy all constraints) are called basic feasible solutions.

The remaining two solutions (solutions 3 and 4) have negative values for some variables and are therefore infeasible.

- We are interested only in feasible solutions and therefore do not evaluate the objective function for infeasible solutions.

Let us consider a non basic solution from the sixth solution. Also assume that variables x_3 and x_4 are fixed to arbitary values (other than zero).

We have to fix them at non negative values because otherwise they will become infeasible.

Let us fix them as $x_3 = 1$ and $x_4 = 1$

On substitution, we get $x_1 + x_2 = 4$ and $3x_1 + 2x_2 = 11$ and get $x_1 = 3, x_2 = 1$ and value of objective function $Z = 23$.

This non-basic feasible solution is clearly inferior to the solution $x_1 = 2, x_2 = 3$ obtained as a basic feasible solution by fixing x_3 and x_4 to zero.

The solution (3, 1) is an interior point in the feasible region while the basic feasible solution (2, 3) is a corner point.

We have already seen that it is enough only to evaluate corner points.

We can observe that the four basic feasible solutions correspond to the four corner points.

Every non basic solution that is feasible corresponds to an interior point in the feasible region and every basic feasible solution corresponds to a corner point solution.

In the algebraic method it is enough only to evaluate the basic solutions, find out the feasible ones and evaluate the objective to obtain the optimal solution.

ALGEBRAIC METHOD

1. Convert the inequalities into equation by adding slack variables
2. Assuming that there are m equations and n variables, set n-m (non basic) variables to zero and evaluate the solution for the remaining m basic variables. Evaluate the objective function if the basic solution is feasible.
3. Perform step 2 for all the ^nC_m combinations of basic variables
4. Identify the optimum solution as the one with the maximum (minimum) value of the objective function.

Limitation

- Evaluates large amount of solutions
- Some are infeasible solutions
- ~~Ite~~ The solution (objective value) is not progressive
- No mechanism to point to best solution (optimum)