Finite Differences

In this chapter we shall discuss about several difference operators such as forward difference operator, backward difference operator, central difference operator, shifting operator etc.

Forward difference operator: The operator Δ is called forward difference operator and defined as,

$$\Delta y_{r-1} = y_r - y_{r-1}$$

where y_r ; $r = 0,1,2, \; \dots \; \dots \; \dots, n$ are values of y .

Backward difference operator: The operator ∇ is called backward difference operator and defined as,

$$\nabla y_r = y_r - y_{r-1}$$

where y_r ; $r = 0,1,2, \; \dots \; \dots \; \dots, n$ are values of y .

Central difference operator: The operator δ is called central difference operator and defined as,

$$\delta y_{(2r-1)/2} = y_r - y_{r-1}$$

where y_r ; $r = 0,1,2, \; \dots \; \dots \; \dots, n$ are values of y .

Shifting operator: The operator E is called shifting operator and defined as,

$$E y_{r-1} = y_r$$

where y_r ; $r = 0,1,2, \; \dots \; \dots \; \dots, n$ are values of y .

Which shows that the effect of E is to shift the functional value of y to it's next higher value.

Averaging operator: The operator μ is called averaging operator and defined as,

$$\mu y_r = \frac{1}{2}\left(y_{(2r+1)/2} - y_{(2r-1)/2} \right)$$

where y_r ; $r = 0,1,2, \; \dots \; \dots \; \dots, n$ are values of y .

Differential operator: The operator D is called differential operator and defined as,

$$D y_r = \frac{d}{dx}\left(y_r \right)$$

Where y_r ; $r = 0,1,2, \; \dots \; \dots \; \dots, n$ are values of y .

Unit operator: The unit operator 1 is defined by,

$$1.y_r = y_r$$

where y_r ; $r = 0,1,2, \; \dots \; \dots \; \dots, n$ are values of y .

Forward Differences: If $y_0, y_1, y_2, \cdots\cdots\cdots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, y_3 - y_2, \cdots\cdots\cdots, y_n - y_{n-1}$ are called the differences of y . If these differences are denoted as follows,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\cdots \; \cdots \; \cdots \; \cdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called the forward difference operator and $\Delta y_0, \; \Delta y_1, \; \cdots\cdots, \; \Delta y_{n-1}$ are called first forward differences.

The second forward differences are,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$\cdots \; \cdots \; \cdots \; \cdots \; \cdots$$
$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

Similarly, we can determine kth forward differences.

$$i.e, \;\; \Delta^k y_{n-1} = \Delta^{k-1} y_n - \Delta^{k-1} y_{n-1}$$

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Backward Differences: If $y_0, y_1, y_2, \cdots\cdots\cdots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1$, $y_3 - y_2, \cdots\cdots\cdots, y_n - y_{n-1}$ are called the differences of y. If these differences are denoted as follows,

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\cdots \; \cdots \; \cdots \; \cdots$$
$$\nabla y_n = y_n - y_{n-1}$$

where ∇ is called the backward difference operator and $\nabla y_1, \nabla y_2, \cdots\cdots, \nabla y_n$ are called first backward differences. The second backward differences are,

$$\nabla^2 y_1 = \nabla y_1 - \nabla y_0$$
$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$
$$\cdots \; \cdots \; \cdots \; \cdots \; \cdots$$
$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

Similarly, we can determine kth backward differences.

$$i.e, \;\; \nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1}$$

Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

Central Differences: If $y_0, y_1, y_2, \cdots\cdots\cdots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1$, $y_3 - y_2, \cdots\cdots\cdots, y_n - y_{n-1}$ are called the differences of y. If these differences are denoted as follows,

$$\delta y_{1/2} = y_1 - y_0$$
$$\delta y_{3/2} = y_2 - y_1$$
$$\cdots\ \cdots\ \cdots\ \cdots$$
$$\delta y_{(2n-1)/2} = y_n - y_{n-1}$$

Where δ is called the central difference operator and $\delta y_{1/2}, \delta y_{3/2}, \cdots\cdots, \delta y_{(2n-1)/2}$ are called first central differences.

The second central differences are,

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$$
$$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots$$
$$\delta^2 y_n = \delta y_{(2n+1)/2} - \delta y_{(2n-1)/2}$$

Similarly, we can determine kth central differences.

$$i.e,\ \delta^k y_n = \delta^{k-1} y_{(2n+1)/2} - \delta^{k-1} y_{(2n-1)/2}$$

Central Difference Table

x	y	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$				
x_2	y_2		$\delta^2 y_2$	$\delta^3 y_{3/2}$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	$\delta^4 y_2$	
x_3	y_3		$\delta^2 y_3$			
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$	$\delta^4 y_3$	$\delta^5 y_{5/2}$
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Interpolation

Interpolation is a numerical technique which is used to estimate unknown values of a function by using known values. For example**,** If we are to find out the population of Bangladesh in 1978 when we know the population of Bangladesh in the year 1971, 1975, 1979, 1984, 1988, 1992 and so on, then the process of finding the population of 1978 is known as interpolation.

Mathematically, let $y = f(x)$ be a function which gives $y_0, y_1, y_2, \cdots\cdots, y_n$ for $x_0, x_1, x_2, \cdots\cdots, x_n$ respectively. The method of finding $f(x)$ for $x = \alpha$ where α lies in the given range is called an interpolation and if α lies outside the given range is called an extrapolation.

Assumption for interpolation: For the application of the methods of interpolation, the following fundamental assumptions are required.

a) In the interval under consideration, the values of the function cannot be jumped or fallen down suddenly.
b) In the absence of any evidence to the contrary, the rise and fall in the values of the function must be uniform.
c) The data can be expressed as a polynomial function so that the method of finite difference be applicable.

Methods of interpolation: The various methods of interpolation are as follows:

a) Method of graph
b) Method of curve fitting
c) Method for finite differences.

In this chapter we shall discuss only interpolation formulae for finite differences. These formulae can be separated as follows:

a) Interpolation formulae for equal intervals
b) Interpolation formulae for unequal intervals
c) Interpolation formulae for central difference.

Interpolation formulae for equal intervals: The interpolation formulae for equal intervals are given bellow:

I. Newton's forward interpolation formula: Suppose, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots\cdots\cdots, (x_n, y_n)$ be a set of $(n+1)$ values of x and y. Let the values of x be equidistant,

$$i.e, \quad x_r = x_0 + rh \qquad ; r = 0, 1, 2, \cdots\cdots, n$$

where h is difference between the points.

Let $y_n(x)$ be a polynomial of nth degree such that y and $y_n(x)$ agree at the tabulated points, which is to be determined. It can be written as,

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots\cdots + a_n(x - x_0)(x - x_1)\cdots\cdots(x - x_{n-1}) \qquad \cdots(1)$$

where the constants a_0, a_1, a_2, … …, a_n can be determine as follows:

Putting $x = x_0$ in Eq.(1) we have,

$$a_0 = y_0$$

Putting $x = x_1$ in Eq.(1) we have,

$$y_1 = a_0 + a_1(x_1 - x_0)$$

$$or, \; y_1 = y_0 + a_1 h$$

$$or, \; a_1 = \frac{y_1 - y_0}{h}$$

$$\therefore \; a_1 = \frac{\Delta y_0}{h}$$

Putting $x = x_2$ in Eq.(1) we have,

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$or, \; y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2h)(h)$$

$$or,\ y_2 = y_0 + 2(y_1 - y_0) + 2a_2h^2$$
$$or,\ y_2 = 2y_1 - y_0 + 2a_2h^2$$
$$or,\ a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2}$$
$$or,\ a_2 = \frac{(y_2 - y_1) - (y_1 - y_0)}{2h^2}$$
$$or,\ a_2 = \frac{\Delta y_1 - \Delta y_0}{2h^2}$$
$$\therefore\ a_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly,

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}$$
$$a_4 = \frac{\Delta^4 y_0}{4!h^4}$$
$$\cdots\cdots\cdots$$
$$a_n = \frac{\Delta^n y_0}{n!h^n}$$

Using these values in Eq.(1) we have,

$$y_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \cdots\cdots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\cdots\cdots(x - x_{n-1}) \qquad \cdots(2)$$

Setting $x = x_0 + ph$ we have,

$$x - x_0 = ph$$
$$x - x_1 = x - x_0 - x_1 + x_0$$
$$= (x - x_0) - (x_1 - x_0)$$
$$= ph - h$$
$$= (p-1)h$$

Similarly, $x - x_2 = (p-2)h$

$$x - x_3 = (p-3)h$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots$$
$$x - x_{n-1} = (p - n + 1)h$$

Equation (2) becomes,

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \cdots\cdots + \frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0 \qquad \cdots(3)$$

This is called Newton's forward interpolation formula.

Note: Newton's forward interpolation formula is used to interpolate the values of *y* near the beginning of a set of tabular values.

II. Newton's backward interpolation formula: Suppose, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots\cdots\cdots, (x_n, y_n)$ be a set of $(n+1)$ values of x and y. Let the values of x be equidistant,

$$i.e, \quad x_r = x_0 + rh \qquad ; r = 0, 1, 2, \cdots\cdots, n$$

where h is difference between the points.

Let $y_n(x)$ be a polynomial of nth degree such that y and $y_n(x)$ agree at the tabulated points, which is to be determined. It can be written as,

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots\cdots + a_n(x - x_n)(x - x_{n-1})\cdots\cdots(x - x_0) \qquad \cdots(1)$$

where the constants $a_0, a_1, a_2, \ldots \ldots, a_n$ can be determine as follows:

Putting $x = x_n$ in Eq.(1) we have,

$$a_0 = y_n$$

Putting $x = x_{n-1}$ in Eq.(1) we have,

$$y_{n-1} = y_n + a_1(x_{n-1} - x_n)$$

$$or, \; y_{n-1} = y_n + a_1(-h)$$

$$or, \; a_1 = \frac{y_n - y_{n-1}}{h}$$

$$\therefore \; a_1 = \frac{\nabla y_n}{h}$$

Putting $x = x_{n-2}$ in Eq.(1) we have,

$$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$or, \; y_{n-2} = y_n + \frac{\nabla y_n}{h}(-2h) + a_2(-2h)(-h)$$

$$or, \; y_{n-2} = y_n - 2(y_n - y_{n-1}) + 2a_2h^2$$

$$or, \; y_{n-2} = -y_n + 2y_{n-1} + 2a_2h^2$$

$$or, \; a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2}$$

$$or, \; a_2 = \frac{(y_n - y_{n-1}) - (y_{n-1} - y_{n-2})}{2h^2}$$

$$or, \; a_2 = \frac{\nabla y_n - \nabla y_{n-1}}{2h^2}$$

$$\therefore \; a_2 = \frac{\nabla^2 y_n}{2!h^2}$$

Similarly,

$$a_3 = \frac{\nabla^3 y_n}{3!h^3}$$

$$a_4 = \frac{\nabla^4 y_n}{4!h^4}$$

$$\cdots\cdots\cdots$$

$$a_n = \frac{\nabla^n y_n}{n!h^n}$$

Using these values in Eq.(1) we have,

$$y_n(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \cdots\cdots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})\cdots\cdots(x - x_1) \qquad \cdots(2)$$

Setting $x = x_n + ph$ we have,

$$x - x_n = ph$$
$$x - x_{n-1} = x - x_n + x_n - x_{n-1}$$
$$= (x - x_n) + (x_n - x_{n-1})$$
$$= ph + h$$
$$= (p+1)h$$

Similarly, $x - x_{n-2} = (p+2)h$

$$x - x_{n-3} = (p+3)h$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$x - x_1 = (p+n-1)h$$

Equation (2) becomes,

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \cdots \cdots + \frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n \qquad \cdots (3)$$

This is called Newton's backward interpolation formula.

Note: Newton's backward interpolation formula is used to interpolate the values of *y* near the end of a set of tabular values.

Problem-01: Construct a difference table to find the polynomial of the data $(1,1),(2,8),(3,27),$ $(4,64),(5,125),(6,216)(7,343),(8,512)$ considering appropriate method. Also find r, where $(9,r)$ is given.

Solution: We may construct any one of forward, backward and central difference tables. Since we also have to find r for $x = 9$ which is nearer at the end of the set of given tabular values, so we will construct the backward difference table.

The backward difference table of the given data is as follows:

x	y	∇	∇^2	∇^3	∇^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		**0**
		127		**6**	
7	343		**42**		
		169			
8	**512**				

This is the required difference table.

Here $x_n = 8,\ h = 1,\ y_n = 512,\ \nabla y_n = 169,\ \nabla^2 y_n = 42,\ \nabla^3 y_n = 6,\ \nabla^4 y_n = 0.$

$$\therefore\ p = \frac{x - x_n}{h} = \frac{x-8}{1} = (x-8)$$

By Newton's backward formula we get,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \cdots \cdots + \frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n$$
$$= 512 + 169(x-8) + \frac{(x-8)(x-8+1)}{2!}\times 42 + \frac{(x-8)(x-8+1)(x-8+2)}{3!}\times 6$$
$$= 512 + 169x - 1352 + 21x^2 - 315x + 1176 + x^3 - 21x^2 + 146x - 336$$
$$= x^3$$

This is the required polynomial.

2nd part: For $x=9$ we get, $y(9)=9^3 \therefore\ r=729$ **(Ans.)**

Problem-02: From the following table of yearly premiums for policies maturing at quinquennial ages, estimate the premiums foe policies maturing at the age of 46 years.

Age(x)	45	50	55	60	65
Premium(y)	2.871	2.404	2.083	1.862	1.712

Solution: Since $x=46$ is nearer at the beginning of the set of given tabular values, so we have to construct the forward difference table.

The forward difference table of the given data is as follows:

Age(x)	Premium(y)	Δ	Δ^2	Δ^3	Δ^4
45	**2.871**				
		-0.467			
50	2.404		**0.146**		
		-0.321		**-0.046**	
55	2.083		0.100		**0.017**
		-0.221		-0.029	
60	1.862		0.071		
		-0.150			
65	1.712				

Here $x=46, h=5,\ x_0=45,\ y_0=45,\ \Delta y_0=-0.467,\ \Delta^2 y_0=0.146,\ \Delta^3 y_0=-0.046,\ \Delta^4 y_0=0.017$

$$\therefore\ p=\frac{x-x_0}{h}=\frac{46-45}{5}=\frac{1}{5}=0.2$$

By Newton's forward formula we get,

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0\cdots\cdots+\frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0$$

$$\therefore\ y(46)=2.871+0.2\times(-0.467)+\frac{0.2(0.2-1)}{2!}\times(0.146)+\frac{0.2(0.2-1)(0.2-2)}{3!}\times(-0.046)+\frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}\times(0.017)$$

$$=2.8710-0.0934-0.01168-0.002208-0.0005712$$
$$=2.763(approx.)$$

Problem-03: The values of $\sin x$ are given below for different values of x, find the value of $\sin 38^0$.

x	15	20	25	30	35	40
$y=\sin x$	0.2588190	0.3420201	0.4226183	0.5	0.5735764	0.6427876

Solution: Since $x=38^0$ is nearer at the end of the set of given tabular values, so we have to construct the backward difference table.

The backward difference table of the given data is as follows:

x	$y=\sin x$	∇	∇^2	∇^3	∇^4	∇^5
15	0.2588190					
		0.0832011				
20	0.3420201		-0.0026029			
		0.0805982		-0.0006136		
25	0.4226183		-0.0032165		0.0000248	
		0.0773817		-0.0005888		**0.0000041**
30	0.5		-0.0038053		**0.0000289**	
		0.0735764		**-0.0005599**		
35	0.5735764		**-0.0043652**			
		0.0692112				
40	**0.6427875**					

Here $x=38,\ x_n=40,\ h=5, y_n=0.6427875,\ \nabla y_n=0.0692112,\ \nabla^2 y_n=-0.0043652,\ \nabla^3 y_n=-0.0005599,$

$\nabla^4 y_n = 0.0000289, \nabla^5 y_n = 0.0000041$.

$$\therefore\ p = \frac{x - x_n}{h} = \frac{38 - 40}{5} = -\frac{2}{5} = -0.4$$

By Newton's backward formula we get,

$$y(38) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \cdots\cdots + \frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n$$

$$= 0.6427876 + (-0.4)\times 0.0692112 + \frac{(-0.4)(-0.4+1)}{2!}\times(-0.0043652) + \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}\times(-0.0005599)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}\times(0.0000289) + \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)}{5!}\times(0.0000041)$$

$$= 0.6427876 - 0.02768448 + 0.00052382 + 0.00003583 - 0.00000120$$

$$= 0.6156614\,(approx.)$$

Problem-04: In an examination the number of candidates who obtained marks between certain limits were as follows:

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Find the number of candidates whose scores lie between 45 and 50.

Solution: First of all we construct a cumulative frequency table for the given data.

Upper limits of the class intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

Since $x = 45$ is nearer at the beginning of the set of values in cumulative frequency table, so we have to construct the forward difference table.

The forward difference table of the given data is as follows:

Marks(x)	Cumulative frequencies(y)	Δ	Δ^2	Δ^3	Δ^4
40	**31**				
		42			
50	73		**9**		
		51		**-25**	
60	124		-16		**37**
		35		12	
70	159		-4		
		31			
80	190				

Here $x = 45,\ x_0 = 40,\ h = 10,\ y_0 = 31,\ \Delta y_0 = 42,\ \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37$.

$$\therefore\ p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = \frac{5}{10} = 0.5$$

By Newton's forward formula we get,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \cdots\cdots + \frac{p(p-1)\cdots\cdots(p-n+1)}{n!}\Delta^n y_0$$

$$\therefore\ y(45) = 31 + 0.5\times 42 + \frac{0.5(0.5-1)}{2!}\times 9 + \frac{0.5(0.5-1)(0.5-2)}{3!}\times(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}\times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4452$$

$$= 47.8673$$

$$= 48\,(approx.)$$

Problem-05: The population of a town in the last six censuses was as given below. Estimate the population for the year 1946.

Year(x)	1911	1921	1931	1941	1951	1961
Population in thousands(y)	12	15	20	27	39	52

Solution: Since $x = 1946$ is nearer at the end of the set of given tabular values, so we have to construct the backward difference table

The backward difference table of the given data is as follows:

$Year(x)$	$Populations(y)$	∇	∇^2	∇^3	∇^4	∇^5
1911	12					
		3				
1921	15		2			
		5		0		
1931	20		2		3	
		7		3		**-10**
1941	27		5		**-7**	
		12		**-4**		
1951	39		**1**			
		13				
1961	**52**					

Here $x = 1946,\ x_n = 1961,\ h = 10,\ y_n = 52,\ \nabla y_n = 13,\ \nabla^2 y_n = 1,\ \nabla^3 y_n = -4, \nabla^4 y_n = -7, \nabla^5 y_n = -10$.

$$\therefore\ p = \frac{x - x_n}{h} = \frac{1946 - 1961}{10} = -\frac{15}{10} = -1.5$$

By Newton's backward formula we get,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \cdots\cdots + \frac{p(p+1)\cdots\cdots(p+n-1)}{n!}\nabla^n y_n$$

$$\therefore\ y(1946) = 52 + (-1.5)\times 13 + \frac{(-1.5)(-1.5+1)}{2!}\times 1 + \frac{(-1.5)(-1.5+1)(-1.5+2)}{3!}\times(-4) + \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)}{4!}\times(-7)$$
$$+\frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5!}\times(-10)$$
$$= 32.3438(approx.)$$

Exercise:

Problem-01: The population of a village in the last six censuses was recorded as follows. Estimate the population for the year 1945.

Year(x)	1941	1951	1961	1971	1981	1991
Population(y)	2500	2800	3200	3700	4350	5225

Problem-02: In a company the number of persons whose daily wage are as follows:

Daily wage in Tk.	0-20	20-40	40-60	60-80	80-100
No. of persons	120	145	200	250	150

Find the number of persons whose daily wage is between TK. 40 and TK.50.

Problem-03: The population of a town in decennial census was recorded as follows. Estimate the population for the year 1985.

Year(x)	1951	1961	1971	1981	1991
Population in thousands(y)	98.752	132.285	168.076	195.690	246.05

Problem-04: The population of a town in decennial census was recorded as follows. Estimate the population for the year 1895.

Year(x)	1891	1901	1911	1921	1931
Population in thousands(y)	46	66	81	93	101

Problem-05: Estimate the production of cotton in the year 1935 from the data given below.

Year(x)	1931	1932	1934	1936	1938
Production in millions of bales(y)	17.1	13.0	14.0	9.6	12.4

Interpolation formulae for unequal intervals: The interpolation formulae for unequal intervals are given bellow:

1) Newton's Interpolation formula for unequal intervals.
2) Lagrange's Interpolation formula for unequal intervals.

Divided Differences: Let $y = f(x)$ be a polynomial which gives $f(x_0)$, $f(x_1)$, $f(x_2), \cdots\cdots, f(x_n)$ at the points x_0, $x_1, x_2, \cdots\cdots, x_n$ (which are not equally spaced)respectively. Then the first divided difference for the arguments x_0 and x_1 is denoted by $f(x_0, x_1)$ or $\Delta f(x)$ and defined as,

$$f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

The second divided difference for the arguments x_0, x_1 and x_2 is denoted by $f(x_0, x_1, x_2)$ or $\Delta^2 f(x)$ and defined as,

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

The third divided difference for the arguments x_0, x_1, x_2 and x_3 is denoted by $f(x_0, x_1, x_2, x_3)$ or $\Delta^3 f(x)$ and defined as,

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_0, x_1, x_2) - f(x_1, x_2, x_3)}{x_0 - x_3}$$

Similarly, the nth divided difference for the arguments $x_0, x_1, x_2, \cdots\cdots, x_n$ is denoted by $f(x_0, x_1, x_2, \cdots\cdots, x_n)$ or $\Delta^n f(x)$ and defined as,

$$f(x_0, x_1, x_2, \cdots\cdots, x_n) = \frac{f(x_0, x_1, \cdots\cdots, x_{n-1}) - f(x_1, x_2, \cdots\cdots, x_n)}{x_0 - x_n}$$

Divided Difference Table

x	$f(x)$	Δ	Δ^2	Δ^3
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Properties of Divided Differences: The properties are given as follows:

1) The divided differences are symmetric. *i.e,* $f(x_0, x_1) = f(x_1, x_0)$.
2) The nth divided differences of a polynomial of the nth degree are constant.
3) The nth divided differences can be expressed as the quotient of two determinants each of order $(n+1)$.

I). **Newton's Interpolation formula for unequal intervals:** Let $y = f(x)$ be a polynomial which gives $f(x_0)$, $f(x_1)$, $f(x_2), \cdots\cdots, f(x_n)$ at the points $x_0, x_1, x_2, \cdots\cdots, x_n$ (which are not equally spaced)respectively. Then the first divided difference for the arguments x and x_0 is given by,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$or, \quad f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \cdots \quad \cdots (1)$$

The second divided difference for the arguments x, x_0 and x_1 is given by,

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1} \quad or, \quad f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1) \quad \cdots \quad \cdots (2)$$

The third divided difference for the arguments x, x_0, x_1 and x_2 is given by,

$$f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$$

$$or,\ f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2) \cdots \cdots (3)$$

The nth divided difference for the arguments x, x_0, $x_1, \cdots \cdots, x_n$ is given by,

$$f(x, x_0, x_1, \cdots \cdots, x_{n-1}) = f(x_0, x_1, \cdots \cdots, x_n) + (x - x_n) f(x, x_0, x_1, \cdots \cdots, x_{n-1}) \cdots \cdots (4)$$

Multiplying Eq.(2) by $(x - x_0)$, Eq.(3) by $(x - x_0)(x - x_1)$ and so on and finally the Eq. (4) by $(x - x_0)(x - x_1) \cdots \cdots (x - x_{n-1})$ and adding with Eq.(1) we get,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$\cdots \cdots + (x - x_0)(x - x_1)(x - x_2) \cdots \cdots (x - x_{n-1}) f(x_0, x_1, x_2, \cdots \cdots x_n) + R_n \cdots \cdots (5)$$

where $R_n = (x - x_0)(x - x_1)(x - x_2) \cdots \cdots (x - x_n) f(x, x_0, x_1, x_2, \cdots \cdots x_n)$

If $f(x)$ be a polynomial of degree n, then the $(n+1)$ th divided difference of $f(x)$ will be zero.

$$\therefore f(x, x_0, x_1, \cdots \cdots, x_n) = 0$$

Then the Eq. (5) can be written as,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$\cdots \cdots + (x - x_0)(x - x_1)(x - x_2) \cdots \cdots (x - x_{n-1}) f(x_0, x_1, x_2, \cdots \cdots x_n) \cdots \cdots (6)$$

This formula is called Newton's divided difference interpolation formula for unequal intervals.

II). **Lagrange's Interpolation formula for unequal intervals:** Let $y = f(x)$ be a polynomial of degree n which gives $f(x_0)$, $f(x_1)$, $f(x_2), \cdots \cdots, f(x_n)$ at the points $x_0, x_1, x_2, \cdots \cdots, x_n$ (which are not equally spaced)respectively. This polynomial can be written as,

$$f(x) = a_0(x - x_1)(x - x_2) \cdots \cdots (x - x_n) + a_1(x - x_0)(x - x_2) \cdots \cdots (x - x_n) + a_2(x - x_0)(x - x_1) \cdots \cdots (x - x_n)$$
$$\cdots \cdots + a_n(x - x_0)(x - x_1) \cdots \cdots (x - x_{n-1}) \qquad \cdots (1)$$

putting $x = x_0$ in Eq.(1) we get,

$$f(x_0) = a_0(x_0 - x_1)(x_0 - x_2) \cdots \cdots (x_0 - x_n)$$

$$or,\ a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \cdots \cdots (x_0 - x_n)}$$

putting $x = x_1$ in Eq.(1) we get,

$$f(x_1) = a_1(x_1 - x_0)(x_1 - x_2) \cdots \cdots (x_1 - x_n)$$

$$or,\ a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \cdots \cdots (x_1 - x_n)}$$

Similarly putting $x = x_2$, $x = x_3, \cdots \cdots x = x_n$ in Eq.(1) we get,

$$a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1) \cdots \cdots (x_2 - x_n)}$$
$$\cdots \cdots \cdots \quad \cdots \cdots \cdots$$
$$a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \cdots \cdots (x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2, \cdots \cdots, a_n$ in Eq.(1) we get,

$$f(x) = \frac{(x - x_1)(x - x_2) \cdots \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots \cdots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) \cdots \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots \cdots (x_1 - x_n)} f(x_1) + \frac{(x - x_0)(x - x_1) \cdots \cdots (x - x_n)}{(x_2 - x_0)(x_2 - x_1) \cdots \cdots (x_2 - x_n)} f(x_2)$$
$$\cdots \cdots + \frac{(x - x_0)(x - x_1) \cdots \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots \cdots (x_n - x_{n-1})} f(x_n) \qquad \cdots (2)$$

This formula is called Lagrange's interpolation formula for unequal intervals.

NOTE: The calculation is more complicated in Lagrange's formula than Newton's formula. The application of the formula is not speedy and there is always a chance of committing some error due to the number of positive and negative signs in the numerator and denominator of each term.

Comparisons between Lagrange and Newton Interpolation:

1). The Lagrange and Newton interpolating formulas provide two different forms for an interpolating polynomial, even though the interpolating polynomial is unique.

2). Lagrange method is numerically unstable but Newton's method is usually numerically stable and computationally efficient.

3). Newton formula is much better for computation than the Lagrange formula.

4). Lagrange form is most often used for deriving formulas for approximating derivatives and integrals

5). Lagrange's form is more efficient then the Newton's formula when you have to interpolate several data sets on the same data points.

Problem-01: Using Newton's divided difference estimate $f(8)$ & $f(15)$ from the following table.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		0
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

Here, $x_0=4,\ x_1=5,\ x_2=7,\ x_3=10,\ x_4=11,\ x_5=13$

$$f(x_0)=48,\ f(x_0,x_1)=52,\ f(x_0,x_1,x_2)=15,\ f(x_0,x_1,x_2,x_3)=1$$

By Newton's divided difference formula we get,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$\cdots\ \cdots+(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{n-1})f(x_0,x_1,x_2,\cdots\cdots x_n)$$
$$=48+(x-4)\times 52+(x-4)(x-5)\times 15+(x-4)(x-5)(x-7)\times 1$$
$$=48+52x-208+15x^2-135x+300+x^3-16x^2+83x-140$$

$\therefore\ f(x)=x^3-x^2$

Now $f(8)=8^3-8^2=512-64=448$ $(Ans.)$

And $f(15)=(15)^3-(15)^2=3375-225=3150$ $(Ans.)$

Problem-02: Using Newton's divided difference estimate $f(x)$ from the following table.

x	-1	0	2	3	4
$f(x)$	-16	-7	-1	8	29

Solution: The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-1	-16				
		$\frac{-7+16}{0+1}=9$			
0	-7		$\frac{3-9}{2+1}=-2$		
		$\frac{-1+7}{2-0}=3$		$\frac{2+2}{3+1}=1$	
2	-1		$\frac{9-3}{3-0}=2$		0
		$\frac{8+1}{3-1}=9$		$\frac{6-2}{4-0}=1$	
3	8		$\frac{21-9}{4-2}=6$		
		$\frac{29-8}{4-3}=21$			
4	29				

Here, $x_0=-1,\ x_1=0,\ x_2=2,\ x_3=3,\ x_4=4$

$$f(x_0)=-16,\ f(x_0,x_1)=9,\ f(x_0,x_1,x_2)=-2,\ f(x_0,x_1,x_2,x_3)=1$$

By Newton's divided difference formula we get,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$\cdots\ \cdots+(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{n-1})f(x_0,x_1,x_2,\cdots\cdots x_n)$$
$$=-16+(x+1)\times 9+x(x+1)\times(-2)+x(x+1)(x-2)\times 1$$
$$=-16+9x+9-2x^2-2x+x^3-2x^2+x^2-2x$$
$$\therefore\ f(x)=x^3-3x^2+5x-7 \quad (Ans.)$$

Problem-03: Using Lagrange's formula estimate $f(x)$ from the following table.

x	0	1	2	5
$f(x)$	2	3	12	147

Solution: Here, $x_0=0,\ x_1=1,\ x_2=2,\ x_3=5$

$$f(x_0)=2,\ f(x_1)=3,\ f(x_2)=12,\ f(x_3)=147$$

By Lagrange's formula we get,

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}f(x_1)+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}f(x_2)$$
$$\cdots\cdots+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}\ f(x_n)$$
$$=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\times 2+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)\cdots\cdots(1-5)}\times 3+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}\times 12+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\times 147$$
$$=\frac{x^3-8x^2+17x-10}{-10}\times 2+\frac{x^3-7x^2+10x}{4}\times 3+\frac{x^3-6x^2+5x}{-6}\times 12+\frac{x^3-3x^2+2x}{60}\times 147$$
$$=\frac{-x^3+8x^2-17x+10}{5}+\frac{3x^3-21x^2+30x}{4}-2x^3+12x^2-10x+\frac{49x^3-147x^2+98x}{20}$$

$$=\frac{1}{20}(-4x^3+32x^2-68x+40+15x^3-105x^2+150x-2x^3+12x^2-10x+49x^3-147x^2+98x)$$

$$=\frac{1}{20}(20x^3+20x^2-20x+40)$$

$$=x^3+x^2-x+2$$

$$\therefore\ f(x)=x^3+x^2-x+2 \quad (Ans.)$$

Problem-04: Using Lagrange's formula estimate $f(10)$ from the following table.

x	5	6	9	11
$f(x)$	12	13	14	16

Solution: Here, $x_0=5,\ x_1=6,\ x_2=9,\ x_3=11$ & $x=10$

$f(x_0)=12,\ f(x_1)=13,\ f(x_2)=14,\ f(x_3)=16$

By Lagrange's formula we get,

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}f(x_1)+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}f(x_2)$$

$$\cdots\cdots+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}f(x_n)$$

$$=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times12+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times13+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(5-11)}\times14+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times16$$

$$=\frac{-4}{-24}\times12+\frac{-5}{15}\times13+\frac{-20}{-24}\times14+\frac{20}{60}\times16$$

$$=2-4.333+11.667+5.333$$

$$=14.667\ \ (Ans.)$$

Problem-05: Using Lagrange's formula estimate $\sqrt[3]{55}$ from the following table.

x	50	52	54	56
$f(x)=\sqrt[3]{x}$	3.684	3.732	3.779	3.825

Solution: Here, $x_0=50,\ x_1=52,\ x_2=54,\ x_3=56$ & $x=55$

$f(x_0)=3.684,\ f(x_1)=3.732,\ f(x_2)=3.779,\ f(x_3)=3.825$

By Lagrange's formula we get,

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}f(x_1)+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)}f(x_2)$$

$$\cdots\cdots+\frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}f(x_n)$$

$$=\frac{(55-52)(55-54)(55-56)}{(50-52)(50-54)(50-56)}\times3.684+\frac{(55-50)(55-54)(55-56)}{(52-50)(52-54)(52-56)}\times3.732+\frac{(55-50)(55-52)(55-56)}{(54-50)(54-52)(54-56)}\times3.779+\frac{(55-50)(55-52)(55-54)}{(56-50)(56-52)(56-54)}\times3.825$$

$$=\frac{-3}{48}\times3.684+\frac{-5}{16}\times3.732+\frac{-15}{-16}\times3.779+\frac{15}{48}\times3.825$$

$$=-0.23025-1.16625+3.5428125+1.1953125$$

$$=3.341625\ (Ans.)$$

Exercise:

Problem-01: Using Lagrange's formula estimate $\sin 39^0$ from the following table.

x	0	10	20	30	40
$f(x)=\sin x$	0	1.1736	0.3420	0.5000	0.6428

Problem-02: Using Lagrange's formula estimate $\log 5.15$ from the following table.

x	5.1	5.2	5.3	5.4	5.5
$f(x)=\log x$	0.7076	0.7160	0.7243	0.7324	0.7404

Problem-03: The following table gives the sales of a concern for the five years. Using Lagrange's formula estimate the sales for the years 1986 &1992.

$Year$	1985	1987	1989	1991	1993
$Sales$	40	43	48	52	57

Problem-04: Using Lagrange's formula estimate $\sqrt{151}$ from the following table.

x	150	152	154	156
$f(x)=\sqrt{x}$	12.247	12.329	12.410	12.490

Problem-05: Using Lagrange's formula estimate $\tan(0.15)$ from the following table.

x	0.10	0.13	0.20	0.22	0.30
$f(x)=\tan x$	0.1003	0.1307	0.2027	0.2236	0.3093

Problem-06: Using Newton's divided difference formula estimate $f(8)$ from the following table.

x	4	5	7	10	11
$f(x)$	48	100	294	900	1210

Problem-07: Using Newton's divided difference formula estimate $f(x)$ from the following table.

x	0	1	4	5
$f(x)$	8	11	68	123

Problem-08: Using Newton's divided difference formula estimate $f(x)$ in powers of $(x-5)$ from the following table.

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

Problem-09: Using Newton's divided difference formula estimate $f(6)$ from the following table.

x	5	7	11	13	21
$f(x)$	150	392	1452	2366	9702

Problem-010: Using Newton's divided difference formula estimate $\tan(0.12)$ from the following table.

x	0.10	0.13	0.20	0.22	0.30
$f(x)=\tan x$	0.1003	0.1307	0.2027	0.2236	0.3093