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FINITE DIFFERENCES OPERATORS

For a function y=f(x), it is given that $y_0, y_1, ..., y_n$ are the values of the variable y corresponding to the equidistant arguments, $x_0, x_1, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h,\ x_3 = x_0 + 3h, ..., x_n = x_0 + nh$. In this case, even though Lagrange and divided difference interpolation polynomials can be used for interpolation, some simpler interpolation formulas can be derived. For this, we have to be familiar with some finite difference operators and finite differences, which were introduced by Sir Isaac Newton. Finite differences deal with the changes that take place in the value of a function f(x) due to finite changes in x. Finite difference operators include, forward difference operator, backward difference operator, shift operator, central difference operator and mean operator.

- **Forward difference operator (Δ) :**

For the values $y_0, y_1, ..., y_n$ of a function y=f(x), for the equidistant values $x_0, x_1, x_2, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h,\ x_3 = x_0 + 3h, ..., x_n = x_0 + nh$, the forward difference operator Δ is defined on the function f(x) as,

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i)$$

That is,

$$\Delta y_i = y_{i+1} - y_i$$

Then, in particular

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0)$$

$$\Rightarrow \qquad \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1 + h) - f(x_1) = f(x_2) - f(x_1)$$

$$\Rightarrow \qquad \Delta y_1 = y_2 - y_1$$

etc.,

$\Delta y_0, \Delta y_1, ..., \Delta y_i, ...$ are known as the **first forward differences.**

The second forward differences are defined as,

$$\Delta^2 f(x_i) = \Delta[\Delta f(x_i)] = \Delta[f(x_i+h) - f(x_i)]$$
$$= \Delta f(x_i+h) - \Delta f(x_i)$$
$$= f(x_i+2h) - f(x_i+h) - [f(x_i+h) - f(x_i)]$$
$$= f(x_i+2h) - 2f(x_i+h) + f(x_i)$$
$$= y_{i+2} - 2y_{i+1} + y_i$$

In particular,

$$\Delta^2 f(x_0) = y_2 - 2y_1 + y_0 \quad or \quad \Delta^2 y_0 = y_2 - 2y_1 + y_0$$

The third forward differences are,

$$\Delta^3 f(x_i) = \Delta[\Delta^2 f(x_i)]$$
$$= \Delta[f(x_i+2h) - 2f(x_i+h) + f(x_i)]$$
$$= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

In particular,

$$\Delta^3 f(x_0) = y_3 - 3y_2 + 3y_1 - y_0 \quad or \quad \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

In general the n^th forward difference,

$$\Delta^n f(x_i) = \Delta^{n-1} f(x_i+h) - \Delta^{n-1} f(x_i)$$

The differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 ....$ are called the **leading differences.**

Forward differences can be written in a tabular form as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	$y_0 = f(x_o)$			
		$\Delta y_0 = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	$y_2 = f(x_2)$		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Example Construct the forward difference table for the following x values and its corresponding f values.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
f	0.003	0.067	0.148	0.248	0.370	0.518	0.697

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

Example Construct the forward difference table, where $f(x) = \frac{1}{x}$, x = 1(0.2)2, 4D.

x	$f(x) = \frac{1}{x}$	Δf first difference	$\Delta^2 f$ second difference	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
1.0	1.000					
		-0.1667				
1.2	0.8333		0.0477			
		-0.1190		-0.0180		
1.4	0.7143		0.0297		0.0082	-0.0045
		-0.0893		-0.0098		
1.6	0.6250		0.0199		0.0037	
		-0.0694		-0.0061		
1.8	0.5556		0.0138			
		-0.0556				
2.0	0.5000					

Example Construct the forward difference table for the data

$$x : -2 \quad 0 \quad 2 \quad 4$$
$$y = f(x) : \quad 4 \quad 9 \quad 17 \quad 22$$

The forward difference table is as follows:

x	y=f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	4			
		Δy_0 =5		
0	9		$\Delta^2 y_0$ =3	
		Δy_1 =8		$\Delta^3 y_0$ =-6
2	17		$\Delta^2 y_1$ =-3	
		Δy_2 =5		
4	22			

Properties of Forward difference operator (Δ):

(i) Forward difference of a constant function is zero.

Proof: Consider the constant function $f(x) = k$

Then, $\Delta f(x) = f(x+h) - f(x) = k - k = 0$

(ii) For the functions $f(x)$ *and* $g(x)$; $\Delta\left(f(x) + g(x)\right) = \Delta f(x) + \Delta g(x)$

Proof: By definition,

$$\begin{aligned}\Delta\left(f(x)+g(x)\right) &= \Delta\left((f+g)(x)\right)\\ &= (f+g)(x+h) - (f+g)(x)\\ &= f(x+h) + g(x+h) - \left(f(x) + g(x)\right)\\ &= f(x+h) - f(x) + g(x+h) - g(x)\\ &= \Delta f(x) + \Delta g(x)\end{aligned}$$

(iii)Proceeding as in (ii), for the constants *a* and *b*,

$$\Delta\left(af(x) + bg(x)\right) = a\Delta f(x) + b\Delta g(x).$$

(iv)Forward difference of the product of two functions is given by,

$$\Delta\left(f(x)g(x)\right) = f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

Proof:

$$\Delta\big(f(x)g(x)\big) = \Delta\big((fg)(x)\big)$$
$$= (fg)(x+h) - (fg)(x)$$
$$= f(x+h)g(x+h) - f(x)g(x)$$

Adding and subtracting $f(x+h)g(x)$, the above gives

$$\Delta\big(f(x)g(x)\big) = f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)$$

$$= f(x+h)\big[g(x+h) - g(x)\big] + g(x)\big[f(x+h) - f(x)\big]$$
$$= f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

Note : Adding and subtracting $g(x+h)f(x)$ instead of $f(x+h)g(x)$, it can also be proved that

$$\Delta\big(f(x)g(x)\big) = g(x+h)\Delta f(x) + f(x)\Delta g(x)$$

(v) Forward difference of the quotient of two functions is given by

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$$

Proof:

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$
$$= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$
$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{g(x)\big[f(x+h) - f(x)\big] - f(x)\big[g(x+h) - g(x)\big]}{g(x+h)g(x)}$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$$

Following are some results on forward differences:

Result 1: The n^{th} forward difference of a polynomial of degree n is constant when the values of the independent variable are at equal intervals.

Result 2: If n is an integer,

$$f(a+nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \cdots + \Delta^n f(a)$$

for the polynomial f(x) in x.

Forward Difference Table

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
x_0	f_0						
x_1	f_1	Δf_0	$\Delta^2 f_0$				
				$\Delta^3 f_0$			
x_2	f_2	Δf_1	$\Delta^2 f_2$		$\Delta^4 f_0$		
				$\Delta^3 f_1$		$\Delta^5 f_0$	
x_3	f_3	Δf_2	$\Delta^2 f_2$		$\Delta^4 f_1$		$\Delta^6 f_0$
				$\Delta^3 f_2$		$\Delta^5 f_1$	
x_4	f_4	Δf_3	$\Delta^2 f_3$		$\Delta^4 f_2$		
				$\Delta^3 f_3$			
x_5	f_5	Δf_4	$\Delta^2 f_4$				
		Δf_5					
x_6	f_6						

Example Express $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the values of the function *f*.

$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = f_2 - f_1 - (f_1 - f_0) = f_2 - 2f_1 + f_0$$

$$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 = \Delta f_2 - \Delta f_1 - (\Delta f_1 - \Delta f_0)$$

$$= (f_3 - f_2) - (f_2 - f_1) - (f_2 - f_1) + (f_1 - f_0)$$

$$= f_3 - 3f_2 + 3f_1 - f_0$$

In general,

$$\Delta^n f_0 = f_n - {}^nC_1 f_{n-1} + {}^nC_2 f_{n-2} - {}^nC_3 f_{n-3} + \ldots + (-1)^n f_0 \quad .$$

If we write y_n to denote f_n the above results takes the following forms:

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} - {}^nC_3\, y_{n-3} + \;.\;.\;.\; + (-1)^n y_0$$

Example Show that the value of y_n can be expressed in terms of the leading value y_0 and the leading differences $\Delta y_0, \Delta^2 y_0, \ldots, \Delta^n y_0$.

Solution

(For notational convenience, we treat y_n as f_n and so on.)

From the forward difference table we have

$$\left.\begin{array}{l} \Delta f_0 = f_1 - f_0 \quad \text{or} \quad f_1 = f_0 + \Delta f_0 \\ \Delta f_1 = f_2 - f_1 \quad \text{or} \quad f_2 = f_1 + \Delta f_1 \\ \Delta f_2 = f_3 - f_2 \quad \text{or} \quad f_3 = f_2 + \Delta f_2 \end{array}\right\}$$

and so on. Similarly,

$$\left.\begin{array}{l} \Delta^2 f_0 = \Delta f_1 - \Delta f_0 \text{ or } \Delta f_1 = \Delta f_0 + \Delta^2 f_0 \\ \Delta^2 f_1 = \Delta f_2 - \Delta f_1 \text{ or } \Delta f_2 = \Delta f_1 + \Delta^2 f_1 \end{array}\right\}$$

and so on. Similarly, we can write

$$\left.\begin{array}{l} \Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 \text{ or } \Delta^2 f_1 = \Delta^2 f_0 + \Delta^3 f_0 \\ \Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1 \text{ or } \Delta^2 f_2 = \Delta^2 f_1 + \Delta^3 f_1 \end{array}\right\}$$

and so on. Also, we can write f_2 as

$$\begin{aligned} f_2 &= (f_0 + \Delta f_0) + (\Delta f_0 + \Delta^2 f_0) \\ &= f_0 + 2\Delta f_0 + \Delta^2 f_0 \\ &= (1+\Delta)^2 f_0 \end{aligned}$$

Hence

$$\begin{aligned} f_3 &= f_2 + \Delta f_2 \\ &= (f_1 + \Delta f_1) + \Delta f_0 + 2\Delta^2 f_0 + \Delta^3 f_0 \\ &= f_0 + 3\Delta f_0 + 3\Delta^2 f_0 + \Delta^3 f_0 \\ &= (1+\Delta)^3 f_0 \end{aligned}$$

That is, we can symbolically write

$$f_1 = (1+\Delta) f_0 \, , \; f_2 = (1+\Delta)^2 f_0 \, , \; f_3 = (1+\Delta)^3 f_0 .$$

Continuing this procedure, we can show, in general

$$f_n = (1+\Delta)^n f_0 .$$

Using binomial expansion, the above is

$$f_n = f_0 + {}^nC_1\Delta f_0 + {}^nC_2\Delta^2 f_0 + \ldots + \Delta^n f_0$$

Thus

$$f_n = \sum_{i=0}^{n} {}^nC_i\, \Delta^i f_0.$$

Backward Difference Operator

For the values $y_0, y_1, \ldots, y_n$ of a function y=f(x), for the equidistant values $x_0, x_1, \ldots, x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h,\ x_3 = x_0 + 3h, \ldots, x_n = x_0 + nh$, the **backward difference operator** ∇ is defined on the function f(x) as,

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = y_i - y_{i-1},$$

which is the **first backward difference.**

In particular, we have the first backward differences,

$$\nabla f(x_1) = y_1 - y_0;\ \nabla f(x_2) = y_2 - y_1\ etc$$

The second backward difference is given by

$$\begin{aligned}\nabla^2 f(x_i) &= \nabla(\nabla f(x_i)) = \nabla\left[f(x_i) - f(x_i - h)\right] = \nabla f(x_i) - \nabla f(x_i - h)\\ &= \left[f(x_i) - f(x_i - h)\right] - \left[f(x_i - h) - f(x_i - 2h)\right]\\ &= (y_i - y_{i-1}) - (y_{i-1} - y_{i-2})\\ &= y_i - 2y_{i-1} + y_{i-2}\end{aligned}$$

Similarly, the third backward difference, $\nabla^3 f(x_i) = y_i - 3y_{i-1} + 3y_{i-2} - y_{i-3}$ and so on.

Backward differences can be written in a tabular form as follows:

x	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_o	$y_0 = f(x_o)$			
		$\nabla y_1 = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	$y_2 = f(x_2)$		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Relation between backward difference and other differences:

1. $\Delta y_0 = y_1 - y_0 = \nabla y_1$; $\Delta^2 y_0 = y_2 - 2y_1 + y_0 = \nabla^2 y_2$ etc.

2. $\Delta - \nabla = \Delta\nabla$

Proof: Consider the function f(x).

$$\Delta f(x) = f(x+h) - f(x)$$
$$\nabla f(x) = f(x) - f(x-h)$$

$$\begin{aligned}(\Delta - \nabla)(f(x)) &= \Delta f(x) - \nabla f(x) \\ &= \left[f(x+h) - f(x)\right] - \left[f(x) - f(x-h)\right] \\ &= \Delta f(x) - \Delta f(x-h) \\ &= \Delta\left[f(x) - f(x-h)\right] \\ &= \Delta\left[\nabla f(x)\right]\end{aligned}$$

$$\Rightarrow \quad \Delta - \nabla = \Delta\nabla$$

3. $\nabla = \Delta E^{-1}$

Proof: Consider the function f(x).

$$\nabla f(x) = f(x) - f(x-h) = \Delta f(x-h) = \Delta E^{-1} f(x) \quad \Rightarrow \quad \nabla = \Delta E^{-1}$$

4. $\nabla = 1 - E^{-1}$

Proof: Consider the function f(x).

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x) = \left(1 - E^{-1}\right) f(x) \quad \Rightarrow \quad \nabla = 1 - E^{-1}$$

Problem: Construct the backward difference table for the data

$$x : -2 \quad 0 \quad 2 \quad 4$$
$$y = f(x) : -8 \quad 3 \quad 1 \quad 12$$

Solution: The backward difference table is as follows:

x	Y=f(x)	∇y	$\nabla^2 y$	$\nabla^3 y$
-2	-8			
		∇y_1 =3-(-8)=11		
0	3		$\nabla^2 y_2$ =-2-11= -13	
		∇y_2 =1-3=-2		$\nabla^3 y_3$ =13-(-13)=26
2	1		$\nabla^2 y_3$ =11-(2)=13	
		∇y_3 =12-1=11		
4	12			

Backward Difference Table

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$	$\nabla^6 f$
x_0	f_0						
x_1	f_1	∇f_1	$\nabla^2 f_2$				
x_2	f_2	∇f_2	$\nabla^2 f_3$	$\nabla^3 f_3$	$\nabla^4 f_4$	$\nabla^5 f_5$	
x_3	f_3	∇f_3	$\nabla^2 f_4$	$\nabla^3 f_4$	$\nabla^4 f_5$		$\nabla^6 f_6$
x_4	f_4	∇f_4	$\nabla^2 f_5$	$\nabla^3 f_5$	$\nabla^4 f_6$	$\nabla^5 f_6$	
x_5	f_5	∇f_5	$\nabla^2 f_6$	$\nabla^3 f_6$			
		∇f_6					
x_6	f_6						

Example Show that any value of f (or y) can be expressed in terms of f_n (or y_n) and its backward differences.

Solution

$\nabla f_n = f_n - f_{n-1}$ implies $f_{n-1} = f_n - \nabla f_n$

and $\nabla f_{n-1} = f_{n-1} - f_{n-2}$ implies $f_{n-2} = f_{n-1} - \nabla f_{n-1}$

$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1}$ implies $\nabla f_{n-1} = \nabla f_n - \nabla^2 f_n$

From equations (1) to (3), we obtain

$$f_{n-2} = f_n - 2\nabla f_n + \nabla^2 f_n .$$

Similarly, we can show that

$$f_{n-3} = f_n - 3\nabla f_n + 3\nabla^2 f_n - \nabla^3 f_n .$$

Symbolically, these results can be rewritten as follows:

$$f_{n-1} = (1-\nabla) f_n \;,\; f_{n-2} = (1-\nabla)^2 f_n \;,\; f_{n-3} = (1-\nabla)^3 f_n .$$

Thus, in general, we can write

$$f_{n-r} = (1-\nabla)^r f_n .$$

i.e., $f_{n-r} = f_n - {}^rC_1 \nabla f_n + {}^rC_2 \nabla^2 f_n - \ . \ . \ . + \ (-1)^r \nabla^r f_n$

If we write y_n to denote f_n the above result is:

$y_{n-r} = y_n - {}^rC_1 \nabla y_n + {}^rC_2 \nabla^2 y_n - \ . \ . \ . + \ (-1)^r \nabla^r y_n$