5 Systems of Equations

5.1 Introduction

In previous chapters we implemented the FVM for a *single* PDE (which was derived from a conservation law). Many phenomena of interest to engineers and scientists are described by *systems* of coupled PDEs (which often arise from several conservation laws). A famous example are the Navier-Stokes equations which, together with the Continuity Equation, form a system of coupled PDEs describing fluid flow. Solving these equations is beyond the scope of an introductory text so we apply the FVM to a simpler system of PDEs which are nonetheless very useful.

5.2 The Shallow Water Equations

After much simplification the Navier-Stokes and Continuity equations can be reduced to the Shallow Water Equations (SWE) which may be expressed in 1D or 2D. The SWE can be regarded as simplified model of water flow and are used by engineers to simulate many phenomena of practical interest including river flooding, tsunami propagation and dam break flows. One key simplification in deriving the SWE is that the flow velocity in the water column is depth-averaged so that the equations model situations where the water velocity does not vary much with depth. We present the SWE in both 1D and 2D since both systems of equations are used extensively by the engineering community.

5.2.1 2D SWE

The 2D SWE form a system of three coupled non-linear hyperbolic PDEs with independent variables t (time) and x, y (*horizontal* space). In differential form (where the partial derivatives are applied to each row) the 2D SWE can be written as the matrix PDE,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = Q \tag{5.1}$$

where,

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} \phi \\ \phi v_x \\ \phi v_y \end{bmatrix}, \tag{5.2a}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \phi v_x \\ \phi v_x^2 + \phi^2/2 \\ \phi v_x v_y \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} = \begin{bmatrix} \phi v_y \\ \phi v_x v_y \\ \phi v_y^2 + \phi^2/2 \end{bmatrix} \tag{5.2b}$$

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ g\,\phi\, b_x \\ g\,\phi\, b_y \end{bmatrix}. \tag{5.2c}$$

$\phi = \phi(t,x,y)$ is called the geopotential and $\phi = g\,h$ where $h = h(t,x,y)$ is the water depth and g is the acceleration due to gravity ($g = 9.81 m/s^2$). $v_x = v_x(t,x,y)$ and $v_y = v_y(t,x,y)$ are the water speeds in x and y directions respectively. U is the column matrix of dependent variables, $F = F(U)$ and $G = G(U)$ are column matrices of fluxes in the x and y directions respectively and Q is a column matrix of source terms which here only includes bathymetric terms where b_x and b_y are bed slopes in the x and y directions (measured positive *downwards*). Note that in some texts U, F, G and Q are referred to as vectors and (5.1) is a vector PDE.

Solving the 2D SWE gives the three components of U (effectively momentum in x and y directions and mass) from which the water depth and flow speeds can be found at times t and points (x, y).

5.2.2 1D SWE

The 1D SWE are an obvious reduction of the 2D SWE to 1D and are used to model thin straight channels. They form a system of two coupled non-linear hyperbolic PDEs with independent variables t (time) and x (*horizontal* space). In differential form (where the partial derivatives are applied to each row) the 1D SWE can be written as the matrix PDE,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = Q \tag{5.3}$$

where,

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \phi \\ \phi v_x \end{bmatrix}, \tag{5.4a}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \phi v_x \\ \phi v_x^2 + \phi^2/2 \end{bmatrix}, \tag{5.4b}$$

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ g\,\phi\, b_x \end{bmatrix}. \tag{5.4c}$$

Variables are defined as in the 2D SWE with the obvious reductions to 1D. Solving the 1D SWE gives the two components of U (effectively mass and momentum in x direction) from which the water depth and flow speed can be found at required times t and points x.

Except for very special situations the SWE do not have analytical solutions. We use the FVM to find approximate solutions. It should be noted that an in-depth treatment of the SWE requires the study of such concepts as supercritical and subcritical flow and Riemann invariants which we leave to a more advanced book to follow. The following in a purely mathematical treatment to show how the FVM applies to a system of PDEs.

5.3 General FVS for the SWE

As we have seen previously, the FVM applies to PDEs that can be written in finite volume form. We will focus on the 2D SWE since the FVM for the 1D SWE is essentially the same. We observe that (5.1) is very similar to (1.1a) so we expect that it will be possible to write (5.1) in finite volume form which we now do.

Let $\underline{H} = F\,\underline{i} + G\,\underline{j}$ and $\underline{v} = v_x\,\underline{i} + v_y\,\underline{j}$ then (5.1) becomes,

$$\frac{\partial U}{\partial t} + \nabla \cdot \underline{H} = Q \tag{5.5a}$$

where,

$$\underline{H} = \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \\ \underline{H}_3 \end{bmatrix} = \begin{bmatrix} \phi \underline{v} \\ \phi v_x \underline{v} + 0.5\phi^2 \underline{i} \\ \phi v_y \underline{v} + 0.5\phi^2 \underline{j} \end{bmatrix} \tag{5.5b}$$

$\underline{H} = \underline{H}(U)$ is the flux density, $\underline{v}$ is the flow velocity and the differential operators are applied to each row of U and $\underline{H}$. (5.5a) represents the SWE written in finite volume form as required. The theory and operations of Chapters 1 and 2 are applied to each row of (5.5a) to give,

$$\frac{\partial \mathrm{U}}{\partial t} = -\frac{1}{A}\oint_{C} \underline{\mathrm{H}}\cdot\underline{\mathrm{n}}\ ds + \mathrm{Q} \tag{5.6}$$

where, as before, U and Q now represent averaged quantities over an arbitrary region of area A in the xy-plane.

Discretising each row of (5.6) in the same way as in Chapters 1 and 2 and rearranging gives,

$$\mathrm{U}_k^{n+1} = \mathrm{U}_k^n - \frac{\Delta t}{A_k}\left(\sum_{sides} \underline{\mathrm{H}}^n \cdot \underline{s}\right) + \Delta t\ \mathrm{Q}_k^n \tag{5.7}$$

Notes:

1. The matrix difference equation (5.7) may be regarded as a general *explicit* FVS for the system of equations in (5.1).
2. Derivation of (5.7) for the system (5.1) is the same as for a single equation written in finite volume form - we simply repeat the steps for each row using the *same* time step Δt.
3. Implementation of (5.7) is done row by row.
4. As in the single equation case, a particular FVS based on (5.7) is constructed by *estimating* the interface fluxes for each row of $\underline{\mathrm{H}}$.
5. Since $\underline{\mathrm{H}} = \underline{\mathrm{H}}(\mathrm{U})$, interface flux estimation may be done by estimating U at cell interfaces or estimating $\underline{\mathrm{H}}$ at cell interfaces directly, in either case using some form of extrapolation.

3. The same time step is used for each row of (5.8).

5.4.2 Lax-Friedrichs Scheme for the SWE

In Chapter 4 we met the Lax-Friedrichs scheme for the 2D linear advection equation on a structured mesh. It is,

$$u_{i,j}^{n+1} = \frac{u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j-1}^{n} + u_{i,j+1}^{n}}{4} - \frac{\Delta t}{A_{i,j}} \sum_{\text{sides}} \underline{H}^{n} \cdot \underline{s} \qquad (5.10)$$

where $\underline{H} = \underline{v}\,u$ and interface fluxes are estimated by linear interpolation. Since the SWE and the linear advection equation have the same finite volume form we expect their equivalent FVS to have the same form. Accordingly the Lax-Friedrichs scheme for the SWE (with Q = 0) is,

$$U_{i,j}^{n+1} = \frac{U_{i-1,j}^{n} + U_{i+1,j}^{n} + U_{i,j-1}^{n} + U_{i,j+1}^{n}}{4} - \frac{\Delta t}{A_{i,j}} \sum_{\text{sides}} \underline{H}^{n} \cdot \underline{s} \qquad (5.11)$$

where U and $\underline{H}$ are given by (5.2a) and (5.5b) respectively and interface fluxes are estimated by linear interpolation as described in Chapter 4.

Notes:

1. As mentioned previously, a FVS is derived from the finite volume form which is independent of the particular PDE or system of PDEs. Hence it is no surprise that a FVS for a single PDE has the same form as its equivalent for a system of PDEs as is the case with the FOU and Lax-Friedrichs schemes.
2. The previous note implies that any FV code to solve a single PDE should be convertible easily to a system of PDEs although there is no guarantee that it will work!
3. (5.11) is an explicit scheme so the time step will be constrained by stability considerations which we now address.

5.5 Heuristic