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E-Lecture-04 / PHS-201 A.

Klein-Gordon Equation:-

$$E^2 = p^2c^2 + m^2c^4$$

$$\Rightarrow E^2 - p^2c^2 = m^2c^4 \quad ; \quad \hat{E} = i\hbar\frac{\partial}{\partial t} \text{ and } \vec{p} = -i\hbar\vec{\nabla}$$

$$\Rightarrow \left(-\hbar^2\frac{\partial^2}{\partial t^2} + \hbar^2c^2\nabla^2\right)\phi = m^2c^4\phi$$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \frac{m^2c^2}{\hbar^2}\phi = 0$$

using $\hbar = c = 1$; one can obtain: $(\partial_\mu\partial^\mu + m^2)\phi = 0$

$$\partial_\mu\partial^\mu = \partial_0\partial^0 - \vec{\nabla}^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \Box$$

Hence $(\Box + m^2)\phi = 0$

For free particle sch. eq^n ($V=0$)

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi \quad \text{—(1)} \quad ; \text{[Using P.E = 0]}$$

↓ complex conjugate

$$-i\hbar\frac{\partial\phi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi^* \quad \text{—(2)}$$

$\therefore$ probability density $= \rho = \phi^*\phi$ —(3)

and current density $j = -\frac{i\hbar}{2m}(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*)$ —(4)

They obey continuity eq^n $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$; [using Eq^n (1) and (2)]

For truely relativistic case ρ as in eq^n (3) should not transform like a scalar, but as a time component of 4-vector whose space component is $\vec{j}$ as given by eq^n (4). Thus $j^\mu = (\rho, \vec{j})$.

$$j^0 = \rho = \frac{i\hbar}{2m}(\phi^*\partial^0\phi - \phi\partial^0\phi^*) = \frac{i\hbar}{2m}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) = \text{Not positive definite.}$$

$$j^i = \vec{j} = \frac{i\hbar}{2m}(\phi^*\partial^i\phi - \phi\partial^i\phi^*) = \frac{i\hbar}{2m}\left(\phi^*\frac{\partial\phi}{\partial x_i} - \phi\frac{\partial\phi^*}{\partial x_i}\right)$$

$$= -\frac{i\hbar}{2m}(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*)$$

Thus: $j^\mu = (\rho, \vec{j}) = \frac{i\hbar}{2m}(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$ —(5)

$$\therefore \partial_\mu j^\mu = \frac{i\hbar}{2m}(\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^*) = \frac{i\hbar}{2m}(\phi^*\Box\phi - \phi\Box\phi^*)$$

$$= \frac{i\hbar}{2m}\left[\phi^*(-m^2\phi) - \phi(-m^2\phi^*)\right] = 0$$

and continuity equation is satisfied.

$E = \pm\sqrt{p^2c^2 + m^2c^4}$ → The negative energy solution is problematic. Here ρ is not positive definite. Hence the interpretation of ϕ as single particle equation is to be abandoned. It should be interpreted as field equation.

There is another problem with energy $E = \pm\sqrt{p^2c^2 + m^2c^4}$. For a free particle, whose energy thereby constant, this difficulty may be avoided, for we may choose the particle to have positive energy and the negative energy states may be ignored. But an interacting particle may exchange energy with its environment and then there would be nothing to stop it cascading down to infinite negative energy states, emitting an infinite amount of energy in the process. This, of course, does not happen and so poses a problem for single particle K-G equation. The interpretation of ϕ, as a quantum field, however clears up this problem.

Dirac Equation :- Since the K-G equation was found physically unsatisfactory, we shall try to conduct a wave equation

$$i\frac{\partial\psi}{\partial t} = \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi = H\psi \quad \text{—(1)} \quad \text{using } \hbar = c = 1$$

where ψ is a column wave function and $\vec{\alpha}, \beta$ are hermitian matrices to make Hamiltonian hermitian, such that a positive conserved probability density exist.

$$-\frac{\partial^2\psi}{\partial t^2} = \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

$$\Rightarrow -\frac{\partial^2\psi}{\partial t^2} = -\sum_{i,j=1}^{3}\left(\frac{\alpha_i\alpha_j + \alpha_j\alpha_i}{2}\right)\frac{\partial^2\psi}{\partial x^i\partial x^j} + \frac{1}{i}\sum_{i=1}^{3}\left(\alpha_i\beta + \beta\alpha_i\right)\frac{\partial\psi}{\partial x^i} + \beta^2m^2\psi$$

$\alpha_i\alpha_j$ may commute so are $\alpha_i\beta$ etc but $\frac{\partial^2}{\partial x^i\partial x^j} = \frac{\partial^2}{\partial x^j\partial x^i}$

$$\left.\begin{aligned} &\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \quad \forall\ i,j = 1,2,3 \\ &\alpha_i\beta + \beta\alpha_i = 0 \quad \text{and} \quad \alpha_i^2 = \beta^2 = 1 \end{aligned}\right\} \text{—(2)}$$

So we have the restrictions ;

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \ ; \ \{\alpha_i, \beta\} = 0 \quad \text{and} \quad \alpha_i^2 = \beta^2 = I.$$

Pauli matrices can only satisfy $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ and all the above relation can not be satisfied by 2x2 matrices alone. The minimum dimension of matrices needed to satisfy the above relations are 4x4 which in turn forces ψ to be a column matrix with 4 rows.

Let us introduce the notation γ^μ :

$$\gamma^0 = \beta \ ; \ \gamma^i = \beta\alpha^i \quad \text{Then} \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad \text{—(3)}$$

Feynmann notation, $\not{a} = a_\mu\gamma^\mu$

$$\{\gamma^i,\gamma^j\} = \beta\alpha^i\beta\alpha^j + \beta\alpha^j\beta\alpha^i \quad \text{as } \beta^2 = I$$
$$= -\alpha^i\beta^2\alpha^j - \alpha^j\beta^2\alpha^i = -(\alpha^i\alpha^j + \alpha^j\alpha^i)$$
$$= -2\delta_{ij}.$$
$$\Rightarrow \{\gamma^i,\gamma^j\} = 2g^{ij} \quad \text{where } g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$
$$\{\gamma^0,\gamma^i\} = 2g^{0i}$$
and $\{\gamma^0,\gamma^0\} = 2g^{00}$

Then the Dirac equation takes the form
$$\beta i\partial_0\psi = \beta(-i\alpha^i\partial_i + \beta m)\psi \quad ; \text{[Multiplying eq}^n \text{ ① with } \beta]$$
$$\Rightarrow (i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi = 0 \quad \text{as } \beta^2 = I$$
$$\Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0 \Rightarrow (i\not{\partial} - m)\psi = 0 \quad \text{——④}$$

In this representation
$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},\ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ and } \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

with $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

also $g^{\mu\nu}\gamma_\mu = \gamma^\nu$ obviously.

Taking hermitian conjugate of ④ we get :
$$\psi^+(i\gamma^{\mu+}\overleftarrow{\partial}_\mu + m) = 0$$

Now
$$\gamma^{0+} = \gamma^0 = \gamma^0\gamma^0\gamma^0 \quad \text{and}$$
$$\vec{\gamma}^+ = (\beta\vec{\alpha})^+ = \vec{\alpha}\beta = \beta(\beta\vec{\alpha})\beta = \gamma^0\vec{\gamma}\gamma^0$$
Again $\vec{\gamma}^+ = (\beta\vec{\alpha})^+ = -\beta\vec{\alpha} = -\vec{\gamma} \rightarrow$ Anti hermitian.
Also $\vec{\gamma}^2 = -I \Rightarrow \gamma^{i2} = -I \Rightarrow \vec{\gamma}^+ = \vec{\gamma}^{-1}$
$$\vec{\gamma}^+\vec{\gamma} = -\vec{\gamma}.\vec{\gamma} = -\vec{\gamma}^2 = I$$

Introducing ; $\bar{\psi} = \psi^+\gamma^0$ and we get
$$\psi^+(i\gamma^{\mu+}\overleftarrow{\partial}_\mu + m) = \psi^+\gamma^0(i\gamma^\mu\gamma^0\overleftarrow{\partial}_\mu + m\gamma^0) = 0 \quad \text{using } \gamma^{0^2} = \beta^2 = I$$
Multiply right by γ^0
$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$$
or, $\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$ ——⑤

Combining ④ and ⑤ $\quad \bar{\psi}(i\overrightarrow{\not{\partial}} - m)\psi + \bar{\psi}(i\overleftarrow{\not{\partial}} + m)\psi = 0$
$$\Rightarrow i(\bar{\psi}\overleftarrow{\not{\partial}}\psi + \bar{\psi}\overrightarrow{\not{\partial}}\psi) = 0 \quad \text{as } i \neq 0 \text{ we have}$$
$$\Rightarrow \bar{\psi}(\overleftarrow{\not{\partial}} + \overrightarrow{\not{\partial}})\psi = 0$$

Continuity eq^n is $\quad \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$

current and probability density together $j^\mu = \bar{\psi}\gamma^\mu\psi = (\rho, \vec{j})$

here $j^0 = \rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$ = positive definite.

Now also $(i\not{\partial} - m)\psi = 0$ and $\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$

$\Rightarrow i\gamma^\mu\partial_\mu\psi - m\psi = 0$ and $i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$

The above two equation imply

$\gamma^\mu\partial_\mu\psi = -im\psi$ and $\partial_\mu\bar{\psi}\gamma^\mu = im\bar{\psi}$

Vector current : $j^\mu = \bar{\psi}\gamma^\mu\psi$

$\partial_\mu j^\mu = \partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi = im\bar{\psi}\psi + \bar{\psi}(-im\psi) = 0$

$\Rightarrow$ vector current j^μ is conserved $\Rightarrow \partial_\mu j^\mu = 0$

But axial vector current $j_5^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi$ where $\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
with $\{\gamma^5, \gamma^\mu\} = 0$

Now $\partial_\mu j_5^\mu = \partial_\mu\bar{\psi}\gamma_5\gamma^\mu\psi + \bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi$
$= -\partial_\mu\bar{\psi}\gamma^\mu\gamma_5\psi + \bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi$
$= -im\bar{\psi}\gamma_5\psi + \bar{\psi}\gamma_5(-im\psi)$
$= -2im\bar{\psi}\gamma_5\psi \neq 0$

Hence axial vector current is not conserved.

Properties of γ^μ, γ^5 :-

Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and with present representation of γ^μ matrices $\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$;

$\{\gamma^5, \gamma^\mu\} = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3)$

Since $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^\mu\gamma^\nu = \begin{cases} \gamma^\nu\gamma^\mu & \text{for } \mu = \nu \\ -\gamma^\nu\gamma^\mu & \text{for } \mu \neq \nu \end{cases}$

we have $\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = -\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3$ because out of four interchange only once there will be $\mu = \nu$ and thrice $\mu \neq \nu$ condition arises. Hence; $\{\gamma^5, \gamma^\mu\} = 0$

and $\gamma^\mu\gamma_\mu = 4$ since $(\gamma^\mu)^2 = -I$ for $\mu = 1, 2, 3$
and $(\gamma^0)^2 = I$

Similarly one can show

$$\begin{cases} \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \end{cases}$$

$\mathrm{Tr}\,\gamma^\mu = 0$; $\mathrm{Tr}\,\gamma^5 = 0$, $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$

$\mathrm{Tr}\,\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n+1}} = 0$

Consider a Lorentz transformation Λ. Let the system is described by ψ in S frame and ψ' in S' frame.

$$\text{(a)}\ i\gamma^\mu \frac{\partial\psi(x)}{\partial x^\mu} - m\psi(x) = 0 \quad \text{and (b)}\ i\gamma^\mu\frac{\partial\psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0 \Big\} \text{ --- (1)}$$

Now $x' = \Lambda x \Rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$ where
$$\Lambda = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
and $\cosh\phi = \gamma$; $\sinh\phi = \beta\gamma$

$\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

But $x'^2 = x^2 \Rightarrow x'_\mu x'^\mu = \Lambda_\mu^{\ \nu} x_\nu \Lambda^\mu_\sigma x^\sigma = x_\nu x^\nu$

$$\Rightarrow \Lambda^{\ \nu}_{\mu}\Lambda^\mu_\sigma = \delta^\nu_\sigma .$$

Also $x = \Lambda^{-1}x' = \Lambda^{-1}\Lambda x \Rightarrow \Lambda\Lambda^{-1} = \Lambda^{-1}\Lambda = I$

$x'_\nu = \Lambda^\mu_\nu x_\mu = \Lambda^\mu_\nu (\Lambda^{-1})^\sigma_\mu x'_\sigma \Rightarrow \Lambda^\mu_\nu(\Lambda^{-1})^\sigma_\mu = \delta^\sigma_\mu$

There must be a local relation between ψ and ψ' so that the observer in second frame may reconstruct ψ' when ψ is given; assume the relation is linear: $\psi'(x') = S(\Lambda)\,\psi(x)$ --- (2)

$S(\Lambda)$ ↳ non singular 4x4 matrix

from eqn (1)(b)

$$i\gamma^\mu \frac{\partial S(\Lambda)\psi(x)}{\partial x'^\mu} - mS\psi(x) = 0$$

or, $i\gamma^\mu S\dfrac{\partial x^\nu}{\partial x'^\mu}\dfrac{\partial\psi(x)}{\partial x^\nu} - mS\psi(x) = 0$; ⓧ S^{-1}

or, $iS^{-1}\gamma^\mu S\dfrac{\partial x^\nu}{\partial x'^\mu}\dfrac{\partial\psi(x)}{\partial x^\nu} - m\psi(x) = 0$

But $x = \Lambda^{-1}x' \Rightarrow x^\nu = (\Lambda^{-1})^\nu_\mu x'^\mu \Rightarrow \dfrac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu_\mu$

$\therefore S^{-1}\gamma^\mu S\dfrac{\partial x^\nu}{\partial x'^\mu}\dfrac{\partial}{\partial x^\nu} = S^{-1}\gamma^\mu S(\Lambda^{-1})^\nu_\mu\dfrac{\partial}{\partial x^\nu}$ | since $\Lambda^\mu_\nu(\Lambda^{-1})^\sigma_\mu = \delta^\sigma_\nu$

Now iff $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu_\nu\gamma^\nu$ then

$$S^{-1}\gamma^\mu S\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \Lambda^\mu_\nu\gamma^\nu(\Lambda^{-1})^\nu_\mu\frac{\partial}{\partial x^\nu} = \delta^\nu_\nu\gamma^\nu\frac{\partial}{\partial x^\nu} = \gamma^\nu\frac{\partial}{\partial x^\nu}$$

and (1)(b) goes over to eqn (1)(a) provided $S^{-1}\gamma^\mu S = \Lambda^\mu_\nu\gamma^\nu$

Since γ^μ and Λ^μ_ν are known, one can find $S(\Lambda)$ from above equation.

— x —