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The free Dirac equation is $(i\gamma^\mu \partial_\mu - m)\psi = 0$

For free particle we seek four momentum eigen-function of Dirac equation of the form $\psi = \underbrace{u(k)}_{\text{four component spinor independent of } x.} e^{-ik\cdot x}$ --- (1)

where $k\cdot x = k_\mu x^\mu = \omega t - \vec{k}\cdot\vec{x}$

we get from the Dirac equation

$$\therefore H u = (\vec{\alpha}\cdot\vec{p} + \beta m) u = E u \quad \text{---} (2)$$

There are 4-independent solution of this equation, two with $E>0$ and two with $E<0$. First take particle at rest, i.e. $\vec{p}=0$. Then

$$H u = \beta m u = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m u = \begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} u = E u$$

and obviously it has eigenvectors

$$\underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\text{degenerate with } E=+m}, \underbrace{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\text{degenerate with } E=-m.} \text{----} (3) \quad \Bigg| \quad \text{Eigenvalues} = +m, +m, -m, -m$$

For $\vec{p} \neq 0$ we have;

$$H u = (\vec{\alpha}\cdot\vec{p} + \beta m) \underbrace{u}_{4\times 1 \text{ matrix}} = \left[\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\vec{p} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m\right] \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

(u_A, u_B: 2×1 matrix)

$$\text{or,} \quad \begin{pmatrix} mI & \vec{\sigma}\cdot\vec{p} \\ +\vec{\sigma}\cdot\vec{p} & -mI \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad \text{---} (4)$$

The above matrix equation reduces to:

$$\left.\begin{aligned} \vec{\sigma}\cdot\vec{p}\, u_B &= (E-m) u_A \\ \vec{\sigma}\cdot\vec{p}\, u_A &= (E+m) u_B \end{aligned}\right\} \text{---} (5)$$

For the two $E>0$ solutions, one can take $u_A^{(s)} = \chi^{(s)}$, where

$$\chi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{---} (6)$$

The corresponding lower components u_B of u are then specified by the second of equation (5):

$$u_B = \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) u_A^{(s)} = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)}$$

and ~~the~~ hence the positive energy four spinor solutions ~~are~~ are

$$u^{(s)} = \begin{pmatrix} u_A^{(s)} \\ u_B^{(s)} \end{pmatrix} = N \begin{pmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{pmatrix} \quad ; \quad E>0 \text{ with } s=1,2$$

N = Normalisation constant

For $E<0$ we take $u_B^{(s)} = \chi^{(s)}$ and then from first eqn of (5) we get

$$u_A^{(s)} = \frac{\vec{\sigma}\cdot\vec{p}}{E-m} u_B^{(s)} = -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^{(s)} \quad ; \quad E = -|E| \text{ for } E<0$$

we obtain $u^{(s+2)} = \begin{pmatrix} u_A^{(s)} \\ u_B^{(s)} \end{pmatrix} = N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$ with $E<0$ with $s=1,2$

For electron with momentum $\vec{p}$, we have four solutions $u^{1,2}$ corresponding to $E>0$ and $u^{(3,4)}$ for $E<0$.

Home work: Verify $u^{(r)\dagger}u^{(s)} = 0 \quad \forall\, r \neq s$ with $r,s = 1,2,3,4$.

III Let us define a operator Σ, as $\Sigma = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \vec{\sigma}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \vec{\sigma}\cdot \underset{4\times4}{I}$

then $\Sigma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

We know, the expectation value of observable 'A' is

$$\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle [A,H]\rangle + \left\langle \frac{\partial A}{\partial t}\right\rangle$$

For the constant of motion $\frac{d\langle A\rangle}{dt} = 0 \Rightarrow [A,H] = 0$ for operator A having no explicit time dependence.

Now $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, $\hat{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, $\hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$H = \hat{\alpha}\cdot\vec{p} + \beta m + V$$

Now $i\hbar \frac{d\Sigma_x}{dt} = \Sigma_x H - H\Sigma_x = \Sigma_x(\vec{\alpha}\cdot\vec{p} + \beta m + V) - (\vec{\alpha}\cdot\vec{p} + \beta m + V)\Sigma_x$

$$\Rightarrow i\hbar\frac{d\Sigma_x}{dt} = \Sigma_x(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m + V) - (\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m + V)\Sigma_x$$

$\Sigma_x = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$ so it obviously commutes with all quantities of the above equation except α_y and α_z.

Hence $i\hbar\frac{d\Sigma_x}{dt} = (\Sigma_x\alpha_y - \alpha_y\Sigma_x)p_y + (\Sigma_x\alpha_z - \alpha_z\Sigma_x)p_z$

Now $[\Sigma_x, \alpha_x] = 0$ and $[\Sigma_x, \beta] = 0$

Now $\Sigma_x\alpha_y - \alpha_y\Sigma_x = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$

$$= \begin{pmatrix} 0 & [\sigma_x,\sigma_y] \\ [\sigma_x,\sigma_y] & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i\sigma_z \\ 2i\sigma_z & 0 \end{pmatrix} = 2i\alpha_z$$

Similarly $\Sigma_x\alpha_z - \alpha_z\Sigma_x = -2i\alpha_y$

$$\therefore\ i\hbar\frac{d\Sigma_x}{dt} = 2i(\alpha_z p_y - \alpha_y p_z) \quad \text{------ (1)}$$

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Again we want to calculate

$$i\hbar \frac{dL_x}{dt} = [L_x, H]$$
$$= L_x(\vec{\alpha}\cdot\vec{p} + \beta m + V) - (\vec{\alpha}\cdot\vec{p} + \beta m + V) L_x$$

L_x commutes with all quantities except P_y and P_z.

$$i\hbar \frac{dL_x}{dt} = \alpha_y [L_x, P_y] + \alpha_z [L_x, P_z]$$

But $[L_x, P_y] = i\hbar P_z$ and $[L_x, P_z] = -i\hbar P_y$.

$$\therefore\ i\hbar \frac{dL_x}{dt} = i\hbar (\alpha_y P_z - \alpha_z P_y) \quad \text{------ (2)}$$

from (1) and (2) we get

$$i\hbar \frac{d}{dt}\left(L_x + \frac{\hbar}{2}\Sigma_x\right) = 0;$$ also for y and z components.

Hence we can write $i\hbar \frac{d}{dt}\left(\vec{L} + \frac{\hbar}{2}\vec{\Sigma}\right) = 0$

Thus $\vec{J} = \vec{L} + \frac{\hbar}{2}\vec{\Sigma}$ commutes with H and is a constant of motion.

ie. $[H, \vec{J}] = 0$

IV Dirac equation for particle with zero rest mass :—

For massive particle we already get

$$u^{(s)} = N\begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{pmatrix}; \text{for } E>0;\ \chi^{(s)} \equiv \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$u^{(s+2)} = N\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \text{ for } E<0;\ s=1,2$$

For $m=0$; therefore

$$\vec{\sigma}\cdot\vec{p}\, u_B = E u_A$$
$$\vec{\sigma}\cdot\vec{p}\, u_A = E u_B$$

Zero rest mass → $E = |\vec{p}|$ as $E^2 = p^2c^2 + m^2c^4$, and $m=0$.

Hence $\hat{n} = \vec{P}/E$ is a unit vector along $\vec{P}$ for $E>0$ and $\hat{n}$ is antiparallel to $\vec{P}$ for $E<0$. Thus.

$$\begin{cases} \vec{\sigma}\cdot\hat{n}\, u_B = u_A \\ \vec{\sigma}\cdot\hat{n}\, u_A = u_B \end{cases}$$

$$\therefore\ u^{(s)} = N\begin{pmatrix} \chi^{s} \\ \vec{\sigma}\cdot\hat{n}\chi^{(s)} \end{pmatrix};\quad E>0,\ s=1,2$$

$$\therefore\ u^{(s+2)} = N\begin{pmatrix} \vec{\sigma}\cdot\hat{n}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix};\quad E<0;\ s=1,2$$

Now $\vec{\sigma}.\hat{n}\, u^{(s)} = N \begin{pmatrix} \vec{\sigma}.\hat{n}\,\chi^{(s)} \\ (\vec{\sigma}.\hat{n})(\vec{\sigma}.\hat{n})\chi^{(s)} \end{pmatrix} = N \begin{pmatrix} \vec{\sigma}.\hat{n}\,\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} = u^{(s+2)}$

in the case of $m=0$, since $(\vec{\sigma}.\hat{n})(\vec{\sigma}.\hat{n}) = \hat{n}.\hat{n} + i\vec{\sigma}.(\hat{n}\times\hat{n}) = 1$

Thus the operator $\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $\vec{\sigma}.\hat{n}$ have same action, because

$$\gamma^5 u^{(s)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} N \begin{pmatrix} \chi^{(s)} \\ \vec{\sigma}.\hat{n}\,\chi^{(s)} \end{pmatrix} = N \begin{pmatrix} \vec{\sigma}.\hat{n}\,\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} = u^{(s+2)}$$

Similarly, $\gamma^5 u^{(s+2)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} N \begin{pmatrix} \vec{\sigma}.\hat{n}\,\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} = N \begin{pmatrix} \chi^{(s)} \\ \vec{\sigma}.\hat{n}\,\chi^{(s)} \end{pmatrix} = u^{(s)}$

The operator $\vec{\sigma}.\hat{n}$ commutes with Dirac Hamiltonian for zero rest mass particle since in this case $H = \hat{\alpha}.\vec{p} = \begin{pmatrix} 0 & \vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & 0 \end{pmatrix} = E \begin{pmatrix} 0 & \vec{\sigma}.\hat{n} \\ \vec{\sigma}.\hat{n} & 0 \end{pmatrix}$ which obviously commutes with $\vec{\sigma}.\hat{n}$. Thus it is a constant of motion for massless particles. Let us write

$$u^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ (\vec{\sigma}.\hat{n})\chi^{(s)} \end{pmatrix} = \begin{pmatrix} \vec{\chi} \\ \vec{\varphi} \end{pmatrix}$$

Instead of $\vec{\chi}$ and $\vec{\varphi}$, one can introduce combination of them:-

$$\vec{\phi} = \tfrac{1}{2}(\vec{\chi} + \vec{\varphi}) = \tfrac{1}{2}(1 + \vec{\sigma}.\hat{n})\,\vec{\chi}$$

and $\vec{F} = \tfrac{1}{2}(\vec{\chi} - \vec{\varphi}) = \tfrac{1}{2}(1 - \vec{\sigma}.\hat{n})\,\vec{\chi}$

The functions ϕ and F have two components and

$$\left.\begin{aligned} \vec{\sigma}.\hat{n}\,\vec{\phi} &= (\vec{\sigma}.\hat{n})\tfrac{1}{2}(1+\vec{\sigma}.\hat{n})\vec{\chi} = \vec{\phi} \\ \vec{\sigma}.\hat{n}\,\vec{F} &= (\vec{\sigma}.\hat{n})\tfrac{1}{2}(1-\vec{\sigma}.\hat{n})\vec{\chi} = -\vec{F} \end{aligned}\right\}$$

Thus $\vec{\phi}$ and $\vec{F}$ are two eigenfunctions of helicity operator $\vec{\sigma}.\hat{n}$ with eigenvalues $+1$ and -1.

Since $\vec{\sigma}.\hat{n}$ has the same action upon the wave function as γ^5 we can write their eigenfunction in the form:

$$\vec{\phi} = \tfrac{1}{2}(1+\gamma^5)\,u^{(s),(s+2)} \quad \text{and} \quad \vec{F} = \tfrac{1}{2}(1-\gamma^5)\,u^{(s),(s+2)}$$

where $1+\gamma^5 = \begin{pmatrix} I & I \\ I & I \end{pmatrix}$

(VI) Dirac Equation and relativistic corrections to the motion of an electron in electromagnetic field:-

Eqn of motion for free particle $\longrightarrow$ Eqn of motion in EM field $(A^0, \vec{A})$

$$\hat{p} \rightarrow \vec{p} - \frac{e\vec{A}}{c} \quad \text{and} \quad E \longrightarrow E - eA_0$$

where $\vec{B} = \vec{\nabla}\times\vec{A}$ and $A^0 = A_0 = \phi$

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Thus $c\vec{\sigma}\cdot\vec{p}\; u_B = (E - mc^2)\, u_A$

$c\vec{\sigma}\cdot\vec{p}\; u_A = (E + mc^2)\, u_B$

goes over to
$$\begin{cases} c\,\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right) u_B = (E - eA_0 - mc^2)\, u_A \\ c\,\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right) u_A = (E - eA_0 + mc^2)\, u_B \end{cases}$$

Let us study the set of equations for the case of non-relativistic motion in a weak field, so that the following assumptions

$$E = E' + mc^2 \;;\quad |E' - eA_0| < mc^2$$

Thus
$$E' u_A = c\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right) u_B + eA_0\, u_A \quad \text{---- (1)}$$
$$(E' + 2mc^2)\, u_B = c\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right) u_A + eA_0\, u_B \quad \text{---- (2)}$$
$$u_B = \frac{c\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right) u_A}{[(E' + 2mc^2) - eA_0]} \simeq \frac{\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right)}{2mc}\, u_A \;; \quad \text{since}$$

$|E' - eA_0| < mc^2$ in the weak field limit.

Substituting this value in (1) we get
$$E' u_A \simeq \frac{\left[c\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right)\right]\left[\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right)\right] u_A}{2mc} + eA_0 u_A$$
$$= \left\{\frac{\left[\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right)\right]^2}{2m} + eA_0\right\} u_A$$

Since $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$

$$\Rightarrow \left[\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right)\right]^2 = \left(\vec{p} - \frac{e\vec{A}}{c}\right)\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right) + i\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right)\times\left(\vec{p} - \frac{e\vec{A}}{c}\right)$$
$$= \left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 + i\vec{\sigma}\cdot\underbrace{\left(i\hbar\vec{\nabla} + \frac{e\vec{A}}{c}\right)\times\left(i\hbar\vec{\nabla} + \frac{e\vec{A}}{c}\right)}_{\Downarrow}$$
$$-\frac{e\hbar}{c}\,\vec{\sigma}\cdot\underbrace{(\vec{\nabla}\times\vec{A} + \vec{A}\times\vec{\nabla})}_{\Downarrow\ \vec{\nabla}\times\vec{A}} \quad \text{H.W}$$

$$\Rightarrow \left[\vec{\sigma}\cdot\left(\vec{p} - \frac{e\vec{A}}{c}\right)\right]^2 = \left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 - \frac{e\hbar}{c}\,\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A})$$

$$\therefore E' u_A = \left[\frac{\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2}{2m} + eA_0 - \frac{e\hbar}{2mc}\,\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A})\right] u_A$$

Now $\vec{\nabla}\times\vec{A} = \vec{B}$, $\frac{e\hbar}{2mc}\vec{\sigma} = \mu_B\vec{\sigma}$ where μ_B = Bhor magneton

Thus the non-relativistic Hamiltonian is
$$H_{NR} = \frac{\left(\vec{P} - \frac{e\vec{A}}{c}\right)^2}{2m} + eA_0 - \vec{\mu}\cdot\vec{B}$$

Here $A_0 = A^0 = \phi$ is the electrostatic potential.

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Spin-orbit interaction:- Let us consider ~~the~~ the motion of a spin $\frac{1}{2}$ particle in an electrostatic field retaining terms upto order of v^2/c^2. For electrostatic field $\vec{A}=0$ and $eA_0 = V(\vec{r})$. Thus

$$[E' - V(r)]u_A = c(\vec{\sigma}\cdot\vec{p})u_B \quad \text{----}(1)$$

$$[E' + 2mc^2 - V(r)]u_B = c(\vec{\sigma}\cdot\vec{p})u_A \quad \text{----}(2)$$

$$\Rightarrow \quad u_B = \frac{c(\vec{\sigma}\cdot\vec{p})}{2mc^2\left[1+\frac{E'-V(r)}{2mc^2}\right]} \simeq \left[1 - \frac{E'-V(r)}{2mc^2}\right]\frac{(\vec{\sigma}\cdot\vec{p})u_A}{2mc}$$; in the weak field limit, where we have used, therefore $|E' - eA_0| = |E' - V(r)| \ll mc^2$.

Using above equation in (1), we get

$$[E' - V(r)]u_A = c(\vec{\sigma}\cdot\vec{p})u_B \cong \left(\frac{\vec{\sigma}\cdot\vec{p}}{2m}\right)\left[1 - \frac{E'-V(r)}{2mc^2}\right](\vec{\sigma}\cdot\vec{p})u_A$$

$$\Rightarrow \left\{\underbrace{\frac{\vec{\sigma}\cdot\vec{p}}{2m}\left[1 - \frac{E'-V(r)}{2mc^2}\right](\vec{\sigma}\cdot\vec{p}) + V(r)}_{H'_{NR}}\right\}u_A = E'u_A \quad \text{----}(3)$$

H.W $(\vec{\sigma}\cdot\vec{p})f(r)(\vec{\sigma}\cdot\vec{p}) = ??$

$$H'_{NR} = \left[1 - \frac{E'-V(r)}{2mc^2}\right]\frac{\hat{p}^2}{2m} + V(r) + \frac{\hbar}{4m^2c^2}\left(\vec{\sigma}\cdot(\vec{\nabla}V\times\hat{p})\right) - \frac{i\hbar}{4m^2c^2}\vec{\nabla}V\cdot\hat{p}$$