Study Material

Dept. of Applied Mathematics with Oceanology and Computer Eng.

Paper No. - MTM 205

Paper Name - Continuum Mechanics

Semester - 2

Topic of Lectures:

How Velocity to be expressed in Lagrangian and eulerian forms.

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Lectures No. 06

Theory of Motion

In Chapter 1 we considered the picture of deformation of a continuum body at two distinct instants of time with emphasis on initial and subsequent configurations of the body without giving any attention to intermediate configuration. This chapter analyses a continuous sequence of configuration with passage of time. We consider the motion of a body in which deformation continually varies with time.

2.1 Time Rate of Change of a Vector and Tensor Properties

Let us now calculate the material time rate of change of a vector property associated with a specific material point of a continuum. When the vector property is expressed in terms of spatial coordinates in spatial description, in the determination of this time rate, one must take account not only of the change at a spatial point but also of the change in the field as observed by the material point due solely to its motion.

Let $\vec{F}$ be any vector property associated with a material point which happens to occupy the spatial position (x_1, x_2, x_3) at time t so that in spatial description we can write

$$\begin{aligned} \vec{F} &= \vec{G}(x_1, x_2, x_3, t) \\ &= \text{a vector function of } (x_1, x_2, x_3, t) \qquad (1) \end{aligned}$$

After an interval of infinitesimal time δt, let this material point move on to a neighbouring position $(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3)$. Let $\vec{F} + \delta\vec{F}$ be the value of the vector property there. Then

$$\vec{F} + \delta\vec{F} = \vec{G}(x_1 + \delta x_1, x_2, + \delta x_2, x_3 + \delta x_3, t + \delta t) \qquad (2)$$

By Taylor's series expansion, we write

$$\begin{aligned} \vec{F} + \delta\vec{F} &= \vec{G}(x_1, x_2, x_3, t) + \frac{\partial \vec{G}}{\partial t}\delta t + \frac{\partial \vec{G}}{\partial x_k}\delta x_k + \ldots \\ &= \vec{F} + \frac{\partial \vec{F}}{\partial t}\delta t + \frac{\partial \vec{F}}{\partial x_k}\delta x_k + \ldots, \text{ using (1)} \end{aligned}$$

The change in $\vec{F}$ in time interval δt is given by

$$\delta \vec{F} = \frac{\partial \vec{F}}{\partial t}\delta t + \frac{\partial \vec{F}}{\partial x_k}\delta x_k + \ldots$$

Therefore the time rate of change of $\vec{F}$ at the instant t is given by

$$\dot{\vec{F}} = \frac{d\vec{F}}{dt} = \lim_{\delta t \to 0}\frac{\delta \vec{F}}{\delta t} = \frac{\partial \vec{F}}{\partial t} + \frac{\partial \vec{F}}{\partial x_k}\cdot\frac{dx_k}{dt}, \text{ other terms vanish as } \delta t \to 0$$

or

$$\frac{d\vec{F}}{dt} = \left(\frac{\partial}{\partial t} + \frac{dx_k}{dt}\cdot\frac{\partial}{\partial x_k}\right)\vec{F} \tag{3}$$

The first term on right hand side of equation (3) represents the rate of change of $\vec{F}$ at a particular location in space, regarded as fixed due to change in time only noted by a fixed observer and is called local rate of change of $\vec{F}$. The second term $\frac{dx_k}{dt}\cdot\frac{\partial \vec{F}}{\partial x_k}$ gives the rate of change of $\vec{F}$ at a particular time due to the movement or convection from one location to another and is known as the convective rate of change of $\vec{F}$ and $\frac{d\vec{F}}{dt} \equiv \frac{\partial \vec{F}}{\partial t} + \frac{dx_k}{dt}\cdot\frac{\partial \vec{F}}{\partial x_k}$ is called the total time rate of change of $\vec{F}$ or substantial rate of change of $\vec{F}$ noted by an observer moving with the continuum.

In material description when vector property $\vec{F}$ associated with a material point is expressed in terms of material coordinates (X_1, X_2, X_3) where (X_1, X_2, X_3) are initial cartesian coordinates of the material point, then

$$\vec{F} = \vec{F}(X_1, X_2, X_3, t) \tag{4}$$

The time rate of change of $\vec{F}$ is given by

$$\frac{d\vec{F}}{dt} = \lim_{\delta t \to 0}\frac{\delta \vec{F}}{\delta t} = \lim_{\delta t \to 0}\frac{\vec{F}(X_1, X_2, X_3, t+\delta t) - \vec{F}(X_1, X_2, X_3, t)}{\delta t}$$

or

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} \tag{5}$$

Similarly, for a scalar property $f = f(x_1, x_2, x_3, t)$,

$$\frac{df}{dt} = \left(\frac{\partial}{\partial t} + \frac{dx_k}{dt}\cdot\frac{\partial}{\partial x_k}\right)f$$

and for a scalar property $f = f(X_1, X_2, X_3, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} \tag{6}$$

Velocity: Velocity of a material point is defined to be the time rate of change of its position. If $\vec{v}$ be the velocity vector of a material point whose position vector is $\vec{r}$ at time t, then

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \tag{7}$$

Let v_i be the components of the velocity of a material point which happens to occupy the spatial position (x_1, x_2, x_3) at time t. Then

$$v_i = \frac{dx_i}{dt} = \dot{x}_i \tag{8}$$

If X_i be the initial coordinates of this material point and u_i are components of displacement, then

$$x_i = X_i + u_i$$

and
$$v_i = \frac{d}{dt}(X_i + u_i) = \frac{du_i}{dt}$$

as X_i is independent of time. Thus velocity of a particle may also be defined as time rate of change of its displacement. If displacement u_i is expressed in material form

$$u_i = u_i(X_1, X_2, X_3, t) \quad \text{and} \quad x_i = x_i(X_1, X_2, X_3, t)$$

then,

$$\left.\begin{aligned} v_i(X_1, X_2, X_3, t) &= \frac{du_i}{dt}(X_1, X_2, X_3, t) = \frac{\partial u_i}{\partial t}(X_1, X_2, X_3) \\ \text{and} \qquad v_i &= \frac{dx_i}{dt}(X_1, X_2, X_3, t) = \frac{\partial x_i}{\partial t} \end{aligned}\right\} \tag{10}$$

If on the other hand the displacement is given in spatial form, then

$$v_i(x_1, x_2, x_3, t) = \frac{d}{dt}[u_i(x_1, x_2, x_3, t)] = \frac{\partial u_i}{\partial t} + u_{i,k}\frac{dx_k}{dt}, \text{ using (6)}$$

or
$$v_i = \frac{\partial u_i}{\partial t} + v_k u_{i,k} = \frac{\partial u_i}{\partial t} + v_j u_{i,j} \text{ using (8)} \tag{11}$$

In equation (11) velocity is given implicitly since it appears as a factor of the second term.

Obs 1. Using equation (8), the material time rate of change of vector property $\vec{F}$, as given in (3), may be written as

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + v_j \vec{F}_{,j} = \frac{\partial \vec{F}}{\partial t} + v_1 \frac{\partial \vec{F}}{\partial x_1} + v_2 \frac{\partial \vec{F}}{\partial x_2} + v_3 \frac{\partial \vec{F}}{\partial x_3}$$

$$= \frac{\partial \vec{F}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{F} \tag{12}$$

The operator $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla})$ is known as differentiation following the motion.

Obs 2. The material time rate of change a tensor property P_{ij} can be written as

$$\frac{dP_{ij}}{dt}(x_1, x_2, x_3, t) = \frac{\partial P_{ij}}{\partial t}(x_1, x_2, x_3, t) + v_k P_{ij,k} \tag{13}$$

Acceleration The acceleration of a material point is defined as the time rate of change of its velocity.

If v_i be components of velocity and f_i components of acceleration, then

$$f_i = \frac{dv_i}{dt} \tag{14}$$

If the velocity is given in material form, then

$$v_i = v_i(X_1, X_2, X_3, t)$$

then,

$$f_i = \frac{d}{dt}[v_i(X_1, X_2, X_3, t)] = \frac{\partial v_i}{\partial t}(X_1, X_2, X_3) = \frac{\partial^2 x_i}{\partial t^2} \tag{15}$$

If on the other hand, the velocity is given in spatial form, then

$$v_i = v_i(x_1, x_2, x_3, t)$$

then,

$$\left.\begin{aligned} f_i &= \frac{dv_i}{dt}(x_1, x_2, x_3, t) = \frac{\partial v_i}{\partial t} + v_{i,j} v_j \\ \text{or} \quad \vec{f} &= \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \end{aligned}\right\} \tag{16}$$

In certain cases it is useful to write (16) in a slightly different manner. We use a vector identity

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}\left(\frac{v^2}{2}\right) - (\vec{v} \cdot \vec{\nabla})\vec{v} \tag{17}$$

$$\therefore \quad \vec{f} = \frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v} \times \text{rot}\,\vec{v} \tag{18}$$