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by

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(Topic: Inner Product Spaces and Hilbert Spaces)

Unit Structure:

1.1 Introduction

The linear space is a generalization of vector space of two and three dimensions. The concept of length of a vector has been introduced in terms norm in a linear space. In a vector space of usual vector one more important notion is there viz the notion of dot product. With the help of dot product the concept of orthogonality can be introduced. This concept of dot product is missing is normed linear space. Hence the question arises whether the dot product and orthogonality can be introduced in arbitrary linear space. In fact, we show in this module that this can be done and thus we define an inner product in a linear space. A linear space equipped with inner product will be called an inner product space. It is shown here that the normed linear space is a special case of inner product space. Though we have first discussed the normed linear space and then inner product space, historically the notion of inner product space was introduced before the notion of normed linear space.

1.2 Objective

We begin with the axiomatic definition of inner product space introduced by the famous mathematician J. Von Neumann. A complete inner product space is called a Hilbert space in the-name of the great German mathematician D. Hilbert. The modern developments in Hilbert spaces are concerned largely with the theory of operators on the spaces. The whole theory was initiated by the work of D. Hilbert (1912) on integral equations. The currently used geometrical notation and terminology is analogous to that the Euclidean geometry. The generalization of the notions of parallelogram law, Pythagorean theorem, Bessel's inequality, Fourier series etc. have been discussed.

1.3 Inner Product Spaces

Def. 1.3.1 *An inner product space is a (complex) vector space V equipped with a mapping from $V \times V$ to $\mathbb{C}$ denoted by $(x, y) \mapsto \langle x, y\rangle$ for all $x, y \in V$ which satisfies the following properties:*

(i) $\langle x, x\rangle \geq 0$.

(ii) $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ *for all* $x, y \in V$.

(iv) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ *for all* $x, y, z \in V$ *and* $\alpha, \beta \in \mathbb{C}$.

$\langle\ ,\ \rangle$ *is called an inner product on* V.

Also for all $x, y, z \in V$ and $\beta \in \mathbb{C}$, we have

$$\begin{aligned} \langle x, y + \beta z\rangle &= \overline{\langle y + \beta z, x\rangle} \text{ [by (iii)]} \\ &= \overline{\langle y, x\rangle} + \overline{\beta}\ \overline{\langle z, x\rangle} \text{ [by (iv)]} \\ &= \langle x, y\rangle + \overline{\beta}\ \langle x, z\rangle. \end{aligned}$$

Example 1.3.1 *(i)* $\mathbb{R}^n$ *is an inner product space with the inner product*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

where $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(ii) $\mathbb{C}^n$ *is an inner product space with the inner product*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

where $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$.

(iii) The sequence space l^2 *is an inner product space with the inner product*

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

where $x = \{x_n\}, y = \{y_n\} \in l^2$.

(iv) For $f, g \in C[0,1]$, define

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

Then $\langle\ ,\ \rangle$ is an inner product on $C[0,1]$.

Def. 1.3.2 *Let $\langle\ ,\ \rangle$ is an inner product on V. For $x \in V$, define*

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Theorem 1.3.1 (Cauchy-Schwarz inequality) *If V is an inner product space and $x, y \in V$, then $|\langle x, y\rangle| \le \|x\|\|y\|$.*

Further, this inequality is an equality if and only if the vectors x and y are linearly dependent.

Proof. For $y = 0$ we have $\langle x, y\rangle = \langle x, 0\rangle = 0$ and $\langle y, y\rangle = \langle 0, 0\rangle = 0$. So, $|\langle x, y\rangle| = 0 = \|x\|\|y\|$ i.e. the inequality holds.

Let $y \neq 0$ and $\alpha \in \mathbb{C}$. Then we have
$0 \le \|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y\rangle = \langle x, y\rangle - \alpha\langle y, x\rangle - \overline{\alpha}\langle x, y\rangle + |\alpha|^2\langle y, y\rangle$.
Let $\alpha = \frac{\langle x, y\rangle}{\|y\|^2}$.
Then, $0 \le \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\|y\|^2} - \frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2$
i.e. $0 \le \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$
i.e. $\|x\|^2\|y\|^2 \ge |\langle x, y\rangle|^2$
i.e. $|\langle x, y\rangle| \le \|x\|\|y\|$.
Now let x and y be linearly dependent. Then $x = ky$ for some $k \in \mathbb{C}$.
So, LHS$= \|ky\|\|y\| = |k|\|y\|^2$ and RHS$= |\langle ky, y\rangle| = |k||\langle y, y\rangle| = |k|\|y\|^2$. Hence the equality holds.
Conversely, if the equality holds in Cauchy-Schwarz inequality then the above computation becomes
$0 = \|x - \alpha y\| \Rightarrow x = \alpha y$ i.e. x and y are linearly dependent.

A linear space V becomes a nls if it is possible to define a norm in V. The same linear space becomes an inner product space if it is possible to define an inner product in it. So the question arises whether a linear space can be both nls as well as inner product space. The following theorem gives the answer.

Theorem 1.3.2 *Every inner product space is a normed space.*

Proof. Let V be an inner product space.

Then for every $x \in V$, we define

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

We will show that this satisfies all axioms of a norm.

We have

(i) $\|x\| = \sqrt{\langle x, x\rangle} \ge 0$ for all $x \in V$.

(ii) $\|x\| = 0 \Leftrightarrow \langle x, x\rangle = 0 \Leftrightarrow x = 0$.

(iii) $\|kx\| = \sqrt{\langle kx, kx\rangle} = \sqrt{k\overline{k}\langle x, x\rangle} = \sqrt{|k|^2\langle x, x\rangle} = |k|\sqrt{\langle x, x\rangle} = |k|\|x\|$ for all $x \in V$ and $k \in \mathbb{C}$.

(iv)

$$\begin{aligned}\|x+y\|^2 &= \langle\, x+y,\ x+y\rangle \\ &= \langle\, x,\ x\rangle + \langle\, x,\ y\rangle + \langle\, y,\ x\rangle + \langle\, y,\ y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle\, x,\ y\rangle + \overline{\langle\, x,\ y\rangle} \\ &= \|x\|^2 + \|y\|^2 + 2Re\langle\, x,\ y\rangle \\ &\le \|x\|^2 + \|y\|^2 + 2\big|\langle\, x,\ y\rangle\big| \\ &\le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \ \text{ (by Cauchy-Schwarz inequality)} \\ &= \big(\|x\| + \|y\|\big)^2\end{aligned}$$

i.e. $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in V$.

Hence every inner product space is a normed linear space with norm defined by $\|x\| = \sqrt{\langle\, x,\ x\rangle}$.
We now define Hilbert space.

Def. 1.3.3 *An inner product space which is complete w.r.t the norm coming out of the inner product is called a Hilbert space.*

Example 1.3.2 (i) $\mathbb{C}^n$ is a finite dimensional Hilbert space w.r.t the inner product given in Example 1.3.1 (*ii*).

(ii) l^2 is a infinite dimensional Hilbert space w.r.t the inner product

$$\langle \alpha, \beta\rangle = \sum_{i=1}^{\infty} \alpha_i \overline{\beta_i}$$

$\forall \alpha = (\alpha_1, \alpha_2, \ldots,), \beta = (\beta_1, \beta_2, \ldots,) \in l^2$.

(iii) For a measure space (X, σ, μ), the space $L^2(\mu)$ is a Hilbert space w.r.t the inner product

$$\langle f, g\rangle = \int fg \ d\mu$$

$\forall f, g \in L^2(\mu)$.

As every inner product space is normed linear space it follows that every complete inner product space is complete normed linear space i.e. every Hilbert space is a Banach space. We have proved that the norm function defined in a normed linear space is a continuous function. In the same manner it is now shown that the inner product function defined in an inner product space is also a continuous function. Thus we have the following theorem.

Theorem 1.3.3 *In an inner product space, the inner product function is a continuous function.*

Proof. Let V be an inner product space and x, y be any elements of V. Let $\{x_n\}$ $\{y_n\}$ sequences in V such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Then we have

$$\begin{aligned}\big|\langle x_n, y_n\rangle - \langle x, y\rangle\big| &= \big|\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle\big| \\ &= \big|\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle\big| \\ &\le \big|\langle x_n, y_n - y\rangle\big| + \big|\langle x_n - x, y\rangle\big| \\ &\le \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \ \text{ (by Cauchy Schwarz inequality)} \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

Hence $\big|\langle x_n, y_n\rangle - \langle x, y\rangle\big| \to 0$ as $n \to \infty$

i.e. $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$.

This shows that inner product function is a continuous function.

Def. 1.3.4 *Two vectors in an inner product space are called orthogonal if* $\langle x, y\rangle = 0$.

A subset E of an inner product space V is called

(i) orthogonal if all the elements in E are pairwise orthogonal i.e. $\langle x, y\rangle = 0\ \forall x, y \in E$.

(ii) is called orthonormal if E is orthogonal and $\|x\| = 1\ \forall x \in E$.

In elementary geometry, we know that the sum of the squares of the sides of a parallelogram is equal to the the sum of the squares of its diagonals i.e. if $ABCD$ is a parallelogram then we have

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$$

i.e.

$$2(AB^2 + BC^2) = AC^2 + BD^2.$$

This law is known as parallelogram law.

This parallelogram law is also true for ordinary vectors as

$$\left|\vec{a} + \vec{b}\right|^2 + \left|\vec{a} - \vec{b}\right|^2 = 2\Big(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\Big).$$

Geometrically this gives

$$\left|\overrightarrow{OC}\right|^2 + \left|\overrightarrow{BA}\right|^2 = 2\Big(\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{OB}\right|^2\Big).$$

Now we state and prove parallelogram law for inner product space.

Theorem 1.3.4 (Parallelogram Law) *Let V be an inner product space and $x, y \in V$. Then* $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|x\|^2)$.

Proof.

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y\rangle + \langle x - y, x - y\rangle \\ &= 2\big(\langle x, x\rangle + \langle y, y\rangle\big) \\ &= 2\big(\|x\|^2 + \|y\|^2\big).\end{aligned}$$

Theorem 1.3.5 *Let V be an inner product space and $x, y \in V$. Then*

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3}\left\|x + i^k y\right\|^2.$$

Proof. We have

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle,\end{aligned}$$

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle,\end{aligned}$$

$$\begin{aligned}\|x+iy\|^2 &= \langle x+iy, x+iy\rangle \\ &= \langle x, x\rangle + \langle x, iy\rangle + \langle iy, x\rangle + \langle iy, iy\rangle \\ &= \|x\|^2 + |i|^2\|y\|^2 - i\langle x, y\rangle + i\langle y, x\rangle \\ &= \|x\|^2 + \|y\|^2 - i\langle x, y\rangle + i\langle y, x\rangle,\end{aligned}$$

and

$$\begin{aligned}\|x-iy\|^2 &= \langle x-iy, x-iy\rangle \\ &= \langle x, x\rangle - \langle x, iy\rangle - \langle iy, x\rangle + \langle iy, iy\rangle \\ &= \|x\|^2 + |i|^2\|y\|^2 + i\langle x, y\rangle - i\langle y, x\rangle \\ &= \|x\|^2 + \|y\|^2 + i\langle x, y\rangle - i\langle y, x\rangle.\end{aligned}$$

Using all the above four equations, we have

$$\begin{aligned}\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle - \|x\|^2 - \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \\ &\quad + i\|x\|^2 + i\|y\|^2 + \langle x, y\rangle - \langle y, x\rangle - i\|x\|^2 - i\|y\|^2 + \langle x, y\rangle - \langle y, x\rangle \\ &= 4\langle x, y\rangle\end{aligned}$$

i.e $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

i.e. $\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3}\left\|x + i^k y\right\|^2$.

This is known as **Polarization identity**.

Theorem 1.3.6 (Gram-Schmidt orthonormalization) *Let $\{x_1, x_2, \ldots,\}$ be a set of linearly independent vectors in an inner product space V.*

Define $y_1 = x_1$, $u_1 = \frac{y_1}{\|y_1\|}$ and for $n = 2, 3, \ldots$, $y_n := x_n - \langle x_n, u_1\rangle u_1 - \langle x_n, u_2\rangle u_2 - \ldots - \langle x_n, u_{n-1}\rangle u_{n-1}$, $u_n = \frac{y_n}{\|y_n\|}$. Then $\{u_1, u_2, \ldots, u_n, \ldots,\}$ is a set of orthonormal vectors and $span\{x_1, x_2, \ldots, x_n\} = span\{u_1, u_2, \ldots, u_n\}$ $\forall n \in \mathbb{N}$.

We have seen that every inner product space is a normed linear space. Is its converse true? Not always. Under certain condition a normed linear space becomes an inner product space, and that condition is the holding of parallelogram law. We now state and prove this theorem.

Theorem 1.3.7 *A Banach space is a Hilbert space if and only if the parallelogram low holds.*

Proof. For simplicity, we consider the Banach space to be real. We suppose that in this Banach space parallelogram law holds. We introduce an inner product in V by

$$\langle x, y\rangle = \frac{1}{4}[\,\|x+y\|^2 - \|x-y\|^2]......................(1)$$

We now show that $\langle , \rangle$ is an inner product on V.

We have from (1), $\langle x, x\rangle = \|x\|^2 \geq 0$.

Also, $\langle y, x\rangle = \frac{1}{4}[\,\|y+x\|^2 - \|y-x\|^2] = \frac{1}{4}[\,\|x+y\|^2 - \|x-y\|^2] = \langle x, y\rangle$.

Now $\langle x, x\rangle = 0 \Leftrightarrow \|x\|^2 = 0 \Leftrightarrow x = 0$.

It remains to show that

$$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$$

$$\text{and } \langle \lambda x, y\rangle = \lambda\langle x, y\rangle \text{where } \lambda \in \mathbb{R}.$$

By parallelogram law, we have

$$\|u+v+w\|^2 + \|u+v-w\|^2 = 2\|u+v\|^2 + 2\|w\|^2$$

$$\text{and } \|u-v+w\|^2 + \|u-v-w\|^2 = 2\|u-v\|^2 + 2\|w\|^2.$$

On substraction, we obtain

$$\|u+v+w\|^2 + \|u+v-w\|^2 - \|u-v+w\|^2 - \|u-v-w\|^2 = 2\|u+v\|^2 - 2\|u-v\|^2$$

Using (1) we get $\langle u+w, v\rangle + \langle u-w, v\rangle = 2\langle u, v\rangle$.

This is true for any $u, v, w \in V$. So taking $w = u$ in the above, we have $\langle 2u, v\rangle + \langle 0, v\rangle = 2\langle u, v\rangle$

or, $\langle 2u, v\rangle = 2\langle u, v\rangle$ [since $\langle 0, v\rangle = \frac{1}{4}[\,\|0+v\|^2 - \|0-v\|^2] = 0$]

Therefore $\langle u+w, v\rangle + \langle u-w, v\rangle = \langle 2u, v\rangle$.

Let x_1, x_2, y be elements in V. Setting $u+w = x_1, u-w = x_2$ and $v = y$ we obtain

$\langle x_1, y\rangle + \langle x_2, y\rangle = \langle x_1 + x_2, y\rangle$........................(2)

We now show that $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ where $\lambda \in \mathbb{R}$...............(3)

In (2), we take $x_1 = x_2 = x$ and obtain

$2\langle x, y\rangle = \langle 2x, y\rangle$.

Now $3\langle x, y\rangle = 2\langle x, y\rangle + \langle x, y\rangle = \langle 2x, y\rangle + \langle x, y\rangle = \langle 2x + x, y\rangle = \langle 3x, y\rangle$,

$4\langle x, y\rangle = 3\langle x, y\rangle + \langle x, y\rangle = \langle 3x, y\rangle + \langle x, y\rangle = \langle 3x + x, y\rangle = \langle 4x, y\rangle$,

and so on.

In general, we thus have by induction that

$n\langle x, y\rangle = \langle nx, y\rangle$ where n is any positive integer.

In (1) taking $-x$ for x we have $\langle -x, y\rangle = \frac{1}{4}[\|-x+y\|^2 - \|-x-y\|^2] = -\frac{1}{4}[\|x+y\|^2 - \|x-y\|^2] = -\langle x, y\rangle$.

If n is any negative integer, let $n = -m$. Then m is a positive integer. Hence

$n\langle x, y\rangle = \langle -mx, y\rangle = -\langle mx, y\rangle = -m\langle x, y\rangle = n\langle x, y\rangle$.

So $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ is true for any integer.

If λ is any rational number, let $\lambda = \frac{p}{q}$.

Then $p\langle x, y\rangle = \langle px, y\rangle = \langle q(\frac{p}{q}x), y\rangle = q\langle \frac{p}{q}x, y\rangle$

i.e. $\frac{p}{q}\langle x, y\rangle = \langle \frac{p}{q}x, y\rangle$.

Thus (3) is true for any rational λ.

Finally, let λ be any real number. Then there exists a sequence r_n of rational numbers such that $r_n \to \lambda$ as $n \to \infty$.

Now for each r_n we have

$r_n\langle x, y\rangle = \langle r_n x, y\rangle$...................................(4)

We have $\left|r_n\langle x, y\rangle - \lambda\langle x, y\rangle\right| = \left|(r_n - \lambda)\langle x, y\rangle\right| = |r_n - \lambda|\,\langle x, y\rangle \to 0$ as $n \to \infty$.

Therefore $r_n\langle x, y\rangle \to \lambda\langle x, y\rangle$ as $n \to \infty$...........................(5)

Again

$$\begin{aligned}\left|\langle r_n x, y\rangle - \langle \lambda x, y\rangle\right| &= \left|\langle (r_n - \lambda)x, y\rangle\right| \\ &= \frac{1}{4}\left|\left\|(r_n - \lambda)x + y\right\|^2 - \left\|(r_n - \lambda)x - y\right\|^2\right| \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

Therefor $\langle r_n x, y\rangle \to \langle \lambda x, y\rangle$ as $n \to \infty$..............................(6)

In (4), letting $n \to \infty$ and using (5) and (6) we get

$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle$ where $\lambda \in \mathbb{R}$.

Conversely if a Banach space is a Hilbert space, then the parallelogram law satisfies here.

This proves the theorem.

Note 1.3.1 *In case the Banach space is complex, then we have to use the Polarization identity to define the inner product of* x, y.

1.4 Orthonormal sets

We now show that the famous Pythagorean theorem is true for an inner product space.

Theorem 1.4.1 *If* $\{x_1, x_2, \ldots, x_n\}$ *is an orthogonal subset of an inner product space* V*, then*

$$\|x_1 + x_2 + \ldots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \ldots + \|x_n\|^2 .$$

Proof. We have

$$
\begin{aligned}
\|x_1 + x_2 + \ldots + x_n\|^2 &= \langle x_1 + x_2 + \ldots + x_n, x_1 + x_2 + \ldots + x_n \rangle \\
&= \langle x_1, x_1\rangle + \langle x_1, x_2\rangle + \ldots \langle x_1, x_n\rangle \\
&+ \langle x_2, x_1\rangle + \langle x_2, x_2\rangle + \ldots \langle x_2, x_n\rangle \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \langle x_n, x_1\rangle + \langle x_n, x_2\rangle + \ldots \langle x_n, x_n\rangle \\
&= \langle x_1, x_1\rangle + 0 + \ldots + 0 \\
&+ 0 + \langle x_2, x_2\rangle + \ldots + 0 \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots \\
&+ 0 + 0 + \ldots + \langle x_n, x_n\rangle [\because \langle x_i, x_j\rangle = 0 \;\text{ for all } i \neq j\;] \\
&= \langle x_1, x_1\rangle + \langle x_2, x_2\rangle + \ldots + \langle x_n, x_n\rangle \\
&= \|x_1\|^2 + \|x_2\|^2 + \ldots + \|x_n\|^2 .
\end{aligned}
$$

Theorem 1.4.2 (Bessel's Inequality) *Let X be an inner product space, $\{u_1, u_2, \ldots,\}$ be a countable orthonormal set in X and $x \in X$. Then*

$$\sum_{n=1}^{\infty} \left|\langle x, u_n\rangle\right| \leq \|x\|^2 ,$$

where equality holds if and only if $x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n$.

Proof. Let $x_m = \sum_{n=1}^{m} \langle x, u_n\rangle u_n$ for $m = 1, 2, \ldots,$.
Since $\{u_1, u_2, \ldots,\}$ is an orthonormal set,

$$
\begin{aligned}
\langle x, x_m\rangle &= \langle x, \sum_{n=1}^{m} \langle x, u_n\rangle u_n \rangle u_n \\
&= \sum_{n=1}^{m} \overline{\langle x, u_n\rangle} \langle x, u_n\rangle \\
&= \sum_{n=1}^{m} \left|\langle x, u_n\rangle\right|^2 ,
\end{aligned}
$$

and $\langle x_m, x\rangle = \sum_{n=1}^{m} \left|\langle x, u_n\rangle\right|^2$.

Similarly, $\langle x_m, x_m\rangle = \sum_{n=1}^{m} \left|\langle x, u_n\rangle\right|^2$.

Now,

$$\begin{aligned} 0 \le \|x - x_m\|^2 &= \langle x - x_m, x - x_m \rangle \\ &= \langle x, x \rangle - \langle x_m, x \rangle - \langle x, x_m \rangle + \langle x_m, x_m \rangle \\ &= \langle x, x \rangle - \sum_{n=1}^{m} \left| \langle x, u_n \rangle \right|^2 \end{aligned}$$

i.e. $\langle x, x \rangle \ge \sum_{n=1}^{m} \left| \langle x, u_n \rangle \right|^2$ for all $m = 1, 2, \ldots,$

i.e. $\langle x, x \rangle \ge \sum_{n=1}^{\infty} \left| \langle x, u_n \rangle \right|^2$.

If the equality holds then by the above result we get

$$\|x - x_m\| \text{ as } m \to \infty$$

$$\text{i.e. } x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n.$$

Conversely, let $x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$. Then

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle \\ &= \langle \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n, \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \rangle \\ &= \lim_{m \to \infty} \sum_{n=1}^{m} \langle x, u_n \rangle \overline{\langle x, u_n \rangle} \\ &= \sum_{n=1}^{\infty} \left| \langle x, u_n \rangle \right|^2. \end{aligned}$$

Theorem 1.4.3 *Let $\{u_1, u_2, \ldots,\}$ be an orthonormal set in an inner product space X and let $k_1, k_2, \ldots, \in \mathbb{C}$.*

(i) If $\sum_{n=1}^{\infty} k_n u_n$ converges to some $x \in X$, then $k_n = \langle x, u_n \rangle \; \forall n$ and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$.

(ii) **(Riesz-Fischer Theorem)** *If X is a Hilbert space and $\sum_{n=1}^{\infty} |k_n|^2 < \infty$, then $\sum_{n=1}^{\infty} k_n u_n$ converges X.*

Proof. (i) First part follows from Bessel's inequality.

If $x = \sum_{n=1}^{\infty} k_n u_n$, then $k_n = \langle x, u_n \rangle \; \forall n$.

Also $\sum_{n=1}^{\infty} |k_n|^2 = \sum_{n=1}^{\infty} \left| \langle x, u_n \rangle \right| = \|x\|^2 < \infty$.

(ii) Let $x_s = \sum_{n=1}^{s} k_n u_n$ for $s = 1, 2, \ldots$.

For $s > m$,

$$\begin{aligned}\|x_s - x_m\|^2 &= \langle \sum_{n=m+1}^{s} k_n u_n, \sum_{j=m+1}^{s} k_j u_j \rangle \\ &= \sum_{n=m+1}^{s} \sum_{j=m+1}^{s} k_n \overline{k_j} \langle u_n, u_j\rangle \\ &= \sum_{n=m+1}^{s} |k_n|^2 .\end{aligned}$$

As $\sum_{n=1}^{\infty} |k_n|^2 < \infty$, $\{x_s\}$ is a Cauchy sequence in the Hilbert space X. Hence $\sum_{n=1}^{\infty} k_n u_n$ converges X.

Def. 1.4.1 (Orthonormal Basis) *An orthonormal set $\{u_\alpha\}_{\alpha\in I}$ in a Hilbert space X is called an orthonormal basis if it is maximal in the sense that if $\{u_\alpha\}_{\alpha\in I}$ is contained in some orthonormal subset $E \subseteq X$, then $E = \{u_\alpha\}_{\alpha\in I}$.*

Theorem 1.4.4 *Every non-zero Hilbert space has an orthonormal basis.*

Theorem 1.4.5 *Let $\{u_\alpha\}_{\alpha\in I}$ be an orthonormal set in an inner product space X and $x \in X$. Then $E_x := \{u_\alpha : \langle x, u_\alpha\rangle \neq 0\}$ is countable, say $\{u_1, u_2, \ldots, \}$.*

Moreover if X is a Hilbert space, then $\sum_{n=1}^{\infty} \langle x, u_n\rangle u_n$ converges in X to some y and $(x - y) \perp u_\alpha \ \forall \alpha \in I$.

Proof. If $x = 0$, then the result holds.
If $x \neq 0$, then define

$$E_j := \{u_\alpha : \|x\| \leq j \ \big|\langle x, u_\alpha\rangle\big|\} \text{ for } j \in \mathbb{N}.$$

Fix j and let $u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_m} \in E_j$.
Then

$$\begin{aligned}0 < m\|x\|^2 &\leq \sum_{i=1}^{m} j^2 \ \big|\langle x, u_{\alpha_i}\rangle\big|^2 \\ &= j^2 \left(\sum_{i=1}^{m} \big|\langle x, u_{\alpha_i}\rangle\big|^2 \right) \\ &\leq j^2 \|x\|^2 \ \text{[By Bessel's inequality]}\end{aligned}$$

So $m \leq j^2$, i.e. E_j contains at most j^2 elements and $E_x = \bigcup_{j=1}^{\infty} E_j$. Thus E_x is countable.

By Bessel's inequality, we have $\sum_{n=1}^{\infty} \big|\langle x, u_n\rangle\big|^2 \leq \|x\|^2 < \infty$.

Now if X is a Hilbert space then by Riesz-Fischer theorem $\sum_{n=1}^{\infty} \langle x, u_n\rangle u_n$ converges to some $y \in X$.

So,

$$\begin{aligned}\langle y, u_\alpha\rangle &= \langle\sum_{n=1}^{\infty}\langle x, u_n\rangle u_n, u_\alpha\rangle\\ &= \sum_{n=1}^{\infty}\langle x, u_n\rangle\langle u_n, u_\alpha\rangle\\ &= \langle x, u_\alpha\rangle\langle u_\alpha, u_\alpha\rangle \text{ if } u_\alpha \in E_x \text{ or } 0 \text{ if } u_\alpha \notin E_x\\ &= \langle x, u_\alpha\rangle.\end{aligned}$$

Thus, $\langle x - y, u_\alpha\rangle = 0 \ \forall\alpha \in I$
i.e. $x - y \perp u_\alpha \ \forall\alpha \in I$.

Theorem 1.4.6 *Let $\{u_\alpha\}_{\alpha\in I}$ be an orthonormal set in a Hilbert space H. then the following are equivalent:*

(i) *$\{u_\alpha\}_{\alpha\in I}$ is an orthonormal basis in H*

(ii) **(Fourier Expansion)** *For each $x \in H$, $x = \sum_{n=1}^{\infty}\langle x, u_n\rangle u_n$*

(iii) **(Parseval's Formula)** *For each $x \in H$, $\|x\|^2 = \sum_{n=1}^{\infty}\left|\langle x, u_n\rangle\right|^2$ where $E_x = \{u_1, u_2, \ldots, \}$.*

Proof. $(i) \Rightarrow (ii)$: Consider $x \in H$. By Theorem 1.4.5, we have that $\sum_{n=1}^{\infty}\langle x, u_n\rangle u_n = y$ for some $y \in H$ and $(x - y) \perp u_\alpha \ \forall\alpha \in I$.

If $x \neq y$, then $u = \frac{x-y}{\|x-y\|}$ is such that $\|u\| = 1$ and $u \perp u_\alpha \ \forall\alpha \in I$, which is a contradiction to the maximality of $\{u_\alpha\}_{\alpha\in I}$ (i.e. orthonormal basis).

So $x = y = \sum_{n=1}^{\infty}\langle x, u_n\rangle u_n$.

$(ii) \Rightarrow (iii)$: Follows from the equality criterion of Theorem 1.4.2.

$(iii) \Rightarrow (i)$: Let E be an orthonormal set in H such that $E \supseteq \{u_\alpha\}_{\alpha\in I}$.

If $\exists u \in E$ such that $u \neq u_\alpha \ \forall\alpha \in I$, then $\langle u, u_\alpha\rangle = 0 \ \forall\alpha \in I$.

By Parseval's formula, $\|u\| = 0$. But $\|u\| = 1$ as $u \in E$. So a contradiction.

1.5 Approximation

Let X be an inner product space, $x \in X$ and $E \subseteq X$. An element $y \in E$ is said to be a best approximation from E to x if

$$\|x - y\| \leq \|x - z\| \ \ \forall z \in E.$$

Theorem 1.5.1 *Let X be an inner product space. If $Erom \subseteq X$ is convex, then there exists one best approximation from E to any $x \in X$.*

Proof. Let y_1, y_2 be two best approximation from the convex set E to x.
Then by parallelogram law,

$$2\|x - y_1\|^2 + 2\|x - y_2\|^2 = \|2x - y_1 - y_2\|^2 + \|y_1 - y_2\|^2 \text{ ....(1)}$$

Let $d = dist(x, E)$. As E is convex, $\frac{y_1+y_2}{2} \in E$.
Therefore $\left\|x - \frac{y_1+y_2}{2}\right\| \geq d$.
So from (1),
$\|y_1 - y_2\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0$
$\Rightarrow y_1 = y_2$.

Theorem 1.5.2 *Let E be a non-empty closed, convex subset of a Hilbert space H. Then for each $x \in H$, there exists a unique best approximation from E to x. In particular, there exists unique element in E with minimal norm.*

Proof. Let $d = dist(x, E)$.
So there exists a sequence $\{y_n\}$ in E such that $\|x - y_n\| \to d$.
By parallelogram law,

$$2\|x - y_n\|^2 + 2\|x - y_m\|^2 = \|2x - y_m - y_n\|^2 + \|y_n - y_m\|^2 \text{ ....(1)}$$

But as E is convex, $\left\|x - \frac{y_n+y_m}{2}\right\| \geq d$.
Then by (1),
$\|y_n - y_m\|^2 \to 2d^2 + 2d^2 - 4d^2 = 0$.
So $\{y_n\}$ is a Cauchy sequence in the Hilbert space H and E is closed.
So $y_n \to y$ in E, i.e. $y \in E$ and $\|x - y\| = \lim_{n\to\infty}\|x - y_n\| = d$.
Uniqueness follows from the previous Theorem.
The statement about minimal norm in the Theorem follows by considering the best approximation from E to 0.

Theorem 1.5.3 *Let F be a subspace of an inner product space X and $x \in X$. Then $y \in F$ is a best approximation to x if and only if $(x - y) \perp F$.*
In that case, $dist(x, F) = \sqrt{\langle x, x - y\rangle}$.

Proof. Let $y \in F$ be such that $(x - y) \perp F$.
Suppose $z \in F$. Then $y - z \in F$.
Now

$$\begin{aligned}\|x - z\|^2 &= \|(x - y) + (y - z)\|^2 \\ &\geq \|x - y\|^2\end{aligned}$$

Hence $\|x - y\| \leq \|x - z\|\ \forall z \in F$.
So y is the best approximation from F to x.
Conversely, let y be the best approximation from F to x.
Fix $z \in F$ with $\|z\| = 1$.
Let $w = y + \langle x - y, z\rangle z \in F$.
Therefore,

$$\begin{aligned}\|x - y\|^2 &\leq \|x - w\|^2 \\ &= \langle x - w, x - w\rangle \\ &= \langle (x - y) - \langle x - y, z\rangle z, (x - y) - \langle x - y, z\rangle z\rangle \\ &= \|x - y\|^2 - \langle x - y, \langle x - y, z\rangle z\rangle - \langle\langle x - y, z\rangle z, x - y\rangle + \left|\langle x - y, z\rangle\right|^2 \langle z, z\rangle \\ &= \|x - y\|^2 - \left|\langle x - y, z\rangle\right|^2 - \left|\langle x - y, z\rangle\right|^2 + \left|\langle x - y, z\rangle\right|^2\end{aligned}$$

i.e. $0 \leq -\big|\langle x-y, z\rangle\big|^2$

i.e. $\langle x-y, z\rangle = 0\ \forall z \in F$. So $(x-y) \perp F$.

2nd Part: Since $y \in F$, $\langle y, x-y\rangle = 0$....................(1)

Hence $dist(x,F) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\langle x, x-y\rangle}$ [by (1)]

Def. 1.5.1 *For a subset E of an inner product space X, we define*

$$E^{\perp} := \{x \in X : \langle x, y\rangle = 0\ \forall y \in E\}.$$

Claim, $E^{\perp}$ is a closed subspace of X.

Let $x, y \in E^{\perp}$. We have to show that $lx + ky \in E^{\perp}$ where $l, k \in \mathbb{C}$.

For $z \in E$, $\langle lx+ky, z\rangle = l\langle x, z\rangle + k\langle y, z\rangle = l.0 + k.0 = 0$

This implies $lx + ky \in E^{\perp}$.

Let $y_n \in E^{\perp}$ be such that $y_n \to y$ for $y \in X$.

For $z \in E$, $\langle y, z\rangle = \lim_{n\to\infty}\langle y_n, z\rangle = 0$.

So, $y \in E^{\perp}$.

Theorem 1.5.4 (Projection Theorem) *Let H be a Hilbert space and F be a closed subspace of H. Then $H = F + F^{\perp}$.*

Moreover $F = F^{\perp\perp}$.

Proof. Let $x \in H$. By Theorem **??**, there exists unique $y \in F$, the best approximation to x.

By Theorem **??**, $x - y \perp F$.

Let $z = x - y$. Then $z \in F^{\perp}$.

So $x = y + z$ where $y \in F$, $z \in F^{\perp}$.

Hence $H = F + F^{\perp}$.

Next, let $x \in F$.

Then $\langle x, y\rangle = 0\ \forall \in F^{\perp}$

$\Rightarrow x \in F^{\perp\perp}$. So $F \subseteq F^{\perp\perp}$.

Now let $x \in F^{\perp\perp}$. As $x \in H$, $\exists y \in F$ and $z \in F^{\perp}$ such that $x = y + z$.

Here $z \in F^{\perp}$ and $z = x - y \in F^{\perp\perp}$ ($\because y \in F \subseteq F^{\perp\perp}$).

So $z = 0$. Hence $x = y + 0 = y \in F$. So $F^{\perp\perp} \subseteq F$.

Therefore, $F = F^{\perp\perp}$.

So given any $x \in H$, $\exists y \in F$, $z \in F^{\perp}$ such that $x = y + z$.

Let $P : H \to F$ be defined by $P(x) = y$.

This linear map is bounded as $\big\|P(x)\big\| = \|y\| \leq \|x\|$ [$\because \|x\|^2 = \|y\|^2 + \|z\|^2$].

So $Ker(P)$ is closed and $Ker(P) = F^{\perp}$. Also $Ran(P) = F$ which is closed.

Further $P^2(x) = P(P(x)) = P(y) = y = P(x)$.

$F^{\perp}$ is called the orthogonal complement of F. The map $P : H \to F$ is called the **orthogonal projection** of H onto F.

Def. 1.5.2 *If a normed space X has the property that for every non-empty closed subspace F of X, there exists a closed subspace G of X such that $X = F + G$, $F \cap G = \{0\}$, then it is said to have the* **complemented subspace property**.

Remark 1.5.1 *Any Banach space which has the complemented subspace property is a Hilbert space.*

Theorem 1.5.5 (Riesz Representation Theorem) *Let H be a Hilbert space and $y \in H$. Then $\phi_y(x) = \langle x, y\rangle \ \forall x \in H$, is a bounded linear functional on H and $\|\phi_y\| = \|y\|$.*
If $\phi \in H^*$, then $\exists$ unique $y \in H$ such that $\phi = \phi_y$.

Proof. $|\phi_y(x)| = |\langle x, y\rangle| \leq \|x\|\|y\| \ \forall x \in H$ and ϕ_y is linear because $\langle , \rangle$ is linear in the first co-ordinate.
So $\|\phi_y\| \leq \|y\|$.
Now, $\phi_y(\frac{y}{\|y\|}) = \langle \frac{y}{\|y\|}, y\rangle = \frac{\|y\|^2}{\|y\|} = \|y\|$.
So, $\|\phi_y\| = \|y\|$.
Let $M = Ker(\phi)$. Hence M is a closed subspace of H.
If $M = H$, then $\phi(x) = 0 \ \forall x \in H$. Thus we have $\phi(x) = 0 = \langle x, 0\rangle \ \forall x \in H$. So $y = 0$ works in this case.
If $M \neq H$, then there exists $x_0 \in H - M$ and $M^\perp \neq 0$.
For this x_0 we have by Theorem 1.5.4 unique $y \in M$ and $z \in M^\perp$ such that $x_0 = y + z$.
Since $z \notin M$, we have $\phi(z) \neq 0$.
Let x be any element of H.
Then we have $\phi\left(x - \frac{\phi(x)}{\phi(z)}z\right) = \phi(x) - \frac{\phi(x)}{\phi(z)}\phi(z) = \phi(x) - \phi(x) = 0$.
Therefore $x - \frac{\phi(x)}{\phi(z)}z \in M$. Since $z \in M^\perp$ we have
$\left\langle x - \frac{\phi(x)}{\phi(z)}z, z\right\rangle = 0$
i.e. $\langle x, z\rangle - \frac{\phi(x)}{\phi(z)}\langle z, z\rangle = 0$
i.e. $\phi(x)\langle z, z\rangle = \phi(z)\langle x, z\rangle$
i.e. $\phi(x) = \frac{\phi(z)}{\|z\|^2} \ \langle x, z\rangle$
i.e. $\phi(x) = \left\langle x, \frac{\overline{\phi(z)}}{\|z\|^2} \ z\right\rangle$
i.e. $\phi(x) = \langle x, y\rangle$ where $y = \frac{\overline{\phi(z)}}{\|z\|^2} \ z$.
Therefore $\phi = \phi_y$ where $y = \frac{\overline{\phi(z)}}{\|z\|^2} \ z$.
If possible let there exists $y_1, y_2 \in H$ be such that $\phi_{y_1} = \phi_{y_2}$.
Now

$$\begin{aligned}
(\phi_{y_1} - \phi_{y_2})(x) &= \phi_{y_1}(x) - \phi_{y_2}(x) \\
&= \langle x, y_1\rangle - \langle x, y_2\rangle \\
&= \langle x, y_1 - y_2\rangle \\
&= \phi_{y_1 - y_2}(x)
\end{aligned}$$

Therefore $\|\phi_{y_1} - \phi_{y_2}\| = \|\phi_{y_1 - y_2}\| = \|y_1 - y_2\| = 0$
i.e. $y_1 = y_2$. Hence y is unique.

1.6 References or Bibliography

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