Topic: Continuity, Product Topology and Metric Topology

Study Material

Paper No. - MTM-206

Paper Name- General Topology

M. Sc. 2nd Semester

Material No. - 1

By

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19. If $A \subset X$, we define the ***boundary*** of A by the equation

$$\text{Bd}\, A = \bar{A} \cap \overline{(X - A)}.$$

(a) Show that Int A and Bd A are disjoint, and $\bar{A} = \text{Int}\, A \cup \text{Bd}\, A$.
(b) Show that Bd $A = \varnothing \Leftrightarrow A$ is both open and closed.
(c) Show that U is open $\Leftrightarrow \text{Bd}\, U = \bar{U} - U$.
(d) If U is open, is it true that $U = \text{Int}(\bar{U})$? Justify your answer.

20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:
(a) $A = \{x \times y \mid y = 0\}$
(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \{x \times y \mid x \text{ is rational}\}$
(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

***21.** (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X - A$ are functions from this collection to itself.
(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.
(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained

§18 Continuous Functions

The concept of continuous function is basic to much of mathematics. Continuous functions on the real line appear in the first pages of any calculus book, and continuous functions in the plane and in space follow not far behind. More general kinds of continuous functions arise as one goes further in mathematics. In this section, we shall formulate a definition of continuity that will include all these as special cases, and we shall study various properties of continuous functions. Many of these properties are direct generalizations of things you learned about continuous functions in calculus and analysis.

Continuity of a Function

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be ***continuous*** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Recall that $f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$; it is empty if V does not intersect the image set $f(X)$ of f.

Continuity of a function depends not only upon the function f itself, but also on the topologies specified for its domain and range. If we wish to emphasize this fact, we can say that f is continuous *relative* to specific topologies on X and Y.

Let us note that if the topology of the range space Y is given by a basis $\mathcal{B}$, then to prove continuity of f it suffices to show that the inverse image of every *basis element* is open. The arbitrary open set V of Y can be written as a union of basis elements

$$V = \bigcup_{\alpha \in J} B_\alpha.$$

Then

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha),$$

so that $f^{-1}(V)$ is open if each set $f^{-1}(B_\alpha)$ is open.

If the topology on Y is given by a subbasis $\mathcal{S}$, to prove continuity of f it will even suffice to show that the inverse image of each *subbasis* element is open. The arbitrary basis element B for Y can be written as a finite intersection $S_1 \cap \cdots \cap S_n$ of subbasis elements; it follows from the equation

$$f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$$

that the inverse image of every basis element is open.

EXAMPLE 1. Let us consider a function like those studied in analysis, a "real-valued function of a real variable,"

$$f : \mathbb{R} \longrightarrow \mathbb{R}.$$

In analysis, one defines continuity of f via the "ϵ-δ definition," a bugaboo over the years for every student of mathematics. As one would expect, the ϵ-δ definition and ours are equivalent. To prove that our definition implies the ϵ-δ definition, for instance, we proceed as follows:

Given x_0 in $\mathbb{R}$, and given $\epsilon > 0$, the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set of the range space $\mathbb{R}$. Therefore, $f^{-1}(V)$ is an open set in the domain space $\mathbb{R}$. Because $f^{-1}(V)$ contains the point x_0, it contains some basis element (a, b) about x_0. We choose δ to be the smaller of the two numbers $x_0 - a$ and $b - x_0$. Then if $|x - x_0| < \delta$, the point x must be in (a, b), so that $f(x) \in V$, and $|f(x) - f(x_0)| < \epsilon$, as desired.

Proving that the ϵ-δ definition implies our definition is no harder; we leave it to you. We shall return to this example when we study metric spaces.

EXAMPLE 2. In calculus one considers the property of continuity for many kinds of functions. For example, one studies functions of the following types:

$f : \mathbb{R} \longrightarrow \mathbb{R}^2$ (curves in the plane)

$f : \mathbb{R} \longrightarrow \mathbb{R}^3$ (curves in space)

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (functions $f(x, y)$ of two real variables)

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (functions $f(x, y, z)$ of three real variables)

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (vector fields $\mathbf{v}(x, y)$ in the plane).

Each of them has a notion of continuity defined for it. Our general definition of continuity includes all these as special cases; this fact will be a consequence of general theorems we shall prove concerning continuous functions on product spaces and on metric spaces.

EXAMPLE 3 Let $\mathbb{R}$ denote the set of real numbers in its usual topology, and let $\mathbb{R}_\ell$ denote the same set in the lower limit topology. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}_\ell$$

be the identity function; $f(x) = x$ for every real number x. Then f is not a continuous function; the inverse image of the open set $[a, b)$ of $\mathbb{R}_\ell$ equals itself, which is not open in $\mathbb{R}$. On the other hand, the identity function

$$g : \mathbb{R}_\ell \longrightarrow \mathbb{R}$$

is continuous, because the inverse image of (a, b) is itself, which is open in $\mathbb{R}_\ell$.

In analysis, one studies several different but equivalent ways of formulating the definition of continuity. Some of these generalize to arbitrary spaces, and they are considered in the theorems that follow. The familiar "ϵ-δ" definition and the "convergent sequence definition" do not generalize to arbitrary spaces; they will be treated when we study metric spaces.

Theorem 18.1. *Let X and Y be topological spaces; let $f : X \rightarrow Y$. Then the following are equivalent:*

(1) *f is continuous.*

(2) *For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.*

(3) *For every closed set B of Y, the set $f^{-1}(B)$ is closed in X*

(4) *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

If the condition in (4) holds for the point x of X, we say that f is ***continuous at the point x***.

Proof. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and that (1) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Assume that f is continuous. Let A be a subset of X. We show that if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set of X containing x; it must intersect A in some point y. Then V intersects $f(A)$ in the point $f(y)$, so that $f(x) \in \overline{f(A)}$, as desired.

(2) $\Rightarrow$ (3). Let B be closed in Y and let $A = f^{-1}(B)$. We wish to prove that A is closed in X; we show that $\bar{A} = A$. By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$. Therefore, if $x \in \bar{A}$,

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B,$$

so that $x \in f^{-1}(B) = A$. Thus $\bar{A} \subset A$, so that $\bar{A} = A$, as desired.

(3) $\Rightarrow$ (1). Let V be an open set of Y. Set $B = Y - V$. Then

$$f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V).$$

Now B is a closed set of Y. Then $f^{-1}(B)$ is closed in X by hypothesis, so that $f^{-1}(V)$ is open in X, as desired.

(1) $\Rightarrow$ (4). Let $x \in X$ and let V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.

(4) $\Rightarrow$ (1). Let V be an open set of Y; let x be a point of $f^{-1}(V)$ Then $f(x) \in V$, so that by hypothesis there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open. ∎

Homeomorphisms

Let X and Y be topological spaces; let $f . X \rightarrow Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \rightarrow X$$

are continuous, then f is called a ***homeomorphism***.

The condition that f^{-1} be continuous says that for each open set U of X, the inverse image of U under the map $f^{-1} : Y \rightarrow X$ is open in Y But the *inverse image* of U under the map f^{-1} is the same as the *image* of U under the map f. See Figure 18.1. So another way to define a homeomorphism is to say that it is a bijective correspondence $f : X \rightarrow Y$ such that $f(U)$ is open if and only if U is open.

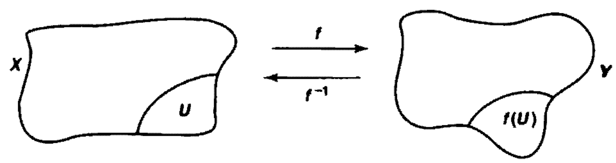

Figure 18.1

This remark shows that a homeomorphism $f : X \rightarrow Y$ gives us a bijective correspondence not only between X and Y but between the collections of open sets of X and of Y. As a result, any property of X that is entirely expressed in terms of the topology of X (that is, in terms of the open sets of X) yields, via the correspondence f, the corresponding property for the space Y. Such a property of X is called a ***topological property*** of X.

You may have studied in modern algebra the notion of an *isomorphism* between algebraic objects such as groups or rings. An isomorphism is a bijective correspondence that preserves the algebraic structure involved. The analogous concept in topology is that of *homeomorphism*; it is a bijective correspondence that preserves the topological structure involved.

Now suppose that $f : X \rightarrow Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \rightarrow Z$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \rightarrow Y$ is a ***topological imbedding***, or simply an ***imbedding***, of X in Y.

EXAMPLE 4. The function $f . \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism See Figure 18 2. If we define $g \cdot \mathbb{R} \to \mathbb{R}$ by the equation

$$g(y) = \frac{1}{3}(y - 1)$$

then one can check easily that $f(g(y)) = y$ and $g(f(x)) = x$ for all real numbers x and y It follows that f is bijective and that $g = f^{-1}$, the continuity of f and g is a familiar result from calculus.

EXAMPLE 5. The function $F . (-1, 1) \to \mathbb{R}$ defined by

$$F(x) = \frac{x}{1 - x^2}$$

is a homeomorphism See Figure 18.3 We have already noted in Example 9 of §3 that F is a bijective order-preserving correspondence; its inverse is the function G defined by

$$G(y) = \frac{2y}{1 + (1 + 4y^2)^{1/2}}.$$

The fact that F is a homeomorphism can be proved in two ways One way is to note that because F is order preserving and bijective, F carnes a basis element for the order topology in $(-1, 1)$ onto a basis element for the order topology in $\mathbb{R}$ and vice versa As a result, F is automatically a homeomorphism of $(-1, 1)$ with $\mathbb{R}$ (both in the order topology) Since the order topology on $(-1, 1)$ and the usual (subspace) topology agree, F is a homeomorphism of $(-1, 1)$ with $\mathbb{R}$

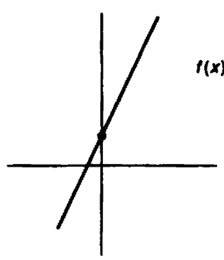

Figure 18.2

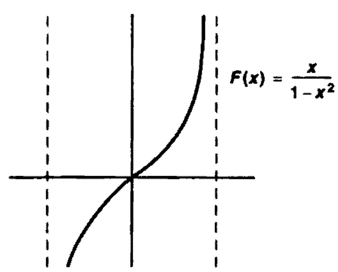

Figure 18.3

A second way to show F a homeomorphism is to use the continuity of the algebraic functions and the square-root function to show that both F and G are continuous These are familiar facts from calculus

EXAMPLE 6 A bijective function $f . X \to Y$ can be continuous without being a homeomorphism One such function is the identity map g $\mathbb{R}_\ell \to \mathbb{R}$ considered in Example 3 Another is the following Let S^1 denote the ***unit circle***,

$$S^1 = \{x \times y \mid x^2 + y^2 = 1\},$$

considered as a subspace of the plane $\mathbb{R}^2$, and let

$$F \cdot [0, 1) \longrightarrow S^1$$

be the map defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. The fact that f is bijective and continuous follows from familiar properties of the trigonometric functions. But the function f^{-1} is not continuous The image under f of the open set $U = [0, \frac{1}{4})$ of the domain, for instance, is not open in S^1, for the point $p = f(0)$ lies in no open set V of $\mathbb{R}^2$ such that $V \cap S^1 \subset f(U)$. See Figure 18.4.

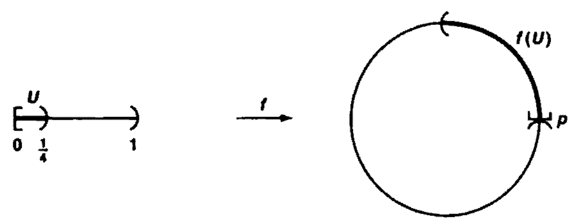

Figure 18.4

EXAMPLE 7. Consider the function

$$g : [0, 1) \longrightarrow \mathbb{R}^2$$

obtained from the function f of the preceding example by expanding the range. The map g is an example of a continuous injective map that is not an imbedding

Constructing Continuous Functions

How does one go about constructing continuous functions from one topological space to another? There are a number of methods used in analysis, of which some generalize to arbitrary topological spaces and others do not. We study first some constructions that do hold for general topological spaces, deferring consideration of the others until later.

Theorem 18.2 (Rules for constructing continuous functions). *Let X, Y, and Z be topological spaces.*

(a) *(Constant function) If $f . X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.*

(b) *(Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.*

(c) *(Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.*

(d) *(Restricting the domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|A : A \to Y$ is continuous.*

(e) *(Restricting or expanding the range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.*

(f) *(Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|U_\alpha$ is continuous for each α.*

Proof. (a) Let $f(x) = y_0$ for every x in X. Let V be open in Y. The set $f^{-1}(V)$ equals X or $\varnothing$, depending on whether V contains y_0 or not. In either case, it is open.

(b) If U is open in X, then $j^{-1}(U) = U \cap A$, which is open in A by definition of the subspace topology.

(c) If U is open in Z, then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X. But

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U),$$

by elementary set theory.

(d) The function $f|A$ equals the composite of the inclusion map $j : A \to X$ and the map $f : X \to Y$, both of which are continuous.

(e) Let $f : X \to Y$ be continuous. If $f(X) \subset Z \subset Y$, we show that the function $g : X \to Z$ obtained from f is continuous. Let B be open in Z. Then $B = Z \cap U$ for some open set U of Y. Because Z contains the entire image set $f(X)$,

$$f^{-1}(U) = g^{-1}(B),$$

by elementary set theory. Since $f^{-1}(U)$ is open, so is $g^{-1}(B)$.

To show $h : X \to Z$ is continuous if Z has Y as a subspace, note that h is the composite of the map $f : X \to Y$ and the inclusion map $j : Y \to Z$.

(f) By hypothesis, we can write X as a union of open sets U_α, such that $f|U_\alpha$ is continuous for each α. Let V be an open set in Y. Then

$$f^{-1}(V) \cap U_\alpha = (f|U_\alpha)^{-1}(V),$$

because both expressions represent the set of those points x lying in U_α for which $f(x) \in V$. Since $f|U_\alpha$ is continuous, this set is open in U_α, and hence open in X. But

$$f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha),$$

so that $f^{-1}(V)$ is also open in X. ∎

Theorem 18.3 (The pasting lemma). *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.*

Proof. Let C be a closed subset of Y. Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C),$$

by elementary set theory. Since f is continuous, $f^{-1}(C)$ is closed in A and, therefore, closed in X. Similarly, $g^{-1}(C)$ is closed in B and therefore closed in X. Their union $h^{-1}(C)$ is thus closed in X. ∎

This theorem also holds if A and B are open sets in X; this is just a special case of the "local formulation of continuity" rule given in preceding theorem.

EXAMPLE 8 Let us define a function $h : \mathbb{R} \to \mathbb{R}$ by setting

$$h(x) = \begin{cases} x & \text{for } x \leq 0, \\ x/2 & \text{for } x \geq 0 \end{cases}$$

Each of the "pieces" of this definition is a continuous function, and they agree on the overlapping part of their domains, which is the one-point set $\{0\}$. Since their domains are closed in $\mathbb{R}$, the function h is continuous. One needs the "pieces" of the function to agree on the overlapping part of their domains in order to have a function at all. The equations

$$k(x) = \begin{cases} x - 2 & \text{for } x \leq 0, \\ x + 2 & \text{for } x \geq 0, \end{cases}$$

for instance, do not define a function On the other hand, one needs some limitations on the sets A and B to guarantee continuity. The equations

$$l(x) = \begin{cases} x - 2 & \text{for } x < 0, \\ x + 2 & \text{for } x \geq 0, \end{cases}$$

for instance, do define a function l mapping $\mathbb{R}$ into $\mathbb{R}$, and both of the pieces are continuous. But l is not continuous; the inverse image of the open set $(1, 3)$, for instance, is the nonopen set $[0, 1)$ See Figure 18.5

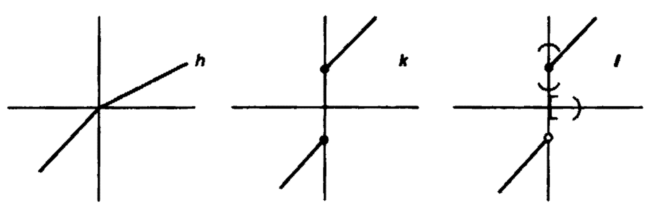

Figure 18.5

Theorem 18.4 (Maps into products). *Let $f : A \to X \times Y$ be given by the equation*

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if the functions

$$f_1 : A \longrightarrow X \quad \text{and} \quad f_2 : A \longrightarrow Y$$

are continuous.

The maps f_1 and f_2 are called the ***coordinate functions*** of f.

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections onto the first and second factors, respectively. These maps are continuous. For $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$, and these sets are open if U and V are open. Note that for each $a \in A$,

$$f_1(a) = \pi_1(f(a)) \quad \text{and} \quad f_2(a) = \pi_2(f(a)).$$

If the function f is continuous, then f_1 and f_2 are composites of continuous functions and therefore continuous. Conversely, suppose that f_1 and f_2 are continuous. We show that for each basis element $U \times V$ for the topology of $X \times Y$, its inverse image $f^{-1}(U \times V)$ is open. A point a is in $f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, that is, if and only if $f_1(a) \in U$ and $f_2(a) \in V$. Therefore,

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V).$$

Since both of the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open, so is their intersection. ∎

There is no useful criterion for the continuity of a map $f : A \times B \to X$ whose *domain* is a product space. One might conjecture that f is continuous if it is continuous "in each variable separately," but this conjecture is not true. (See Exercise 12.)

EXAMPLE 9 In calculus, a *parametrized curve* in the plane is defined to be a continuous map $f : [a, b] \to \mathbb{R}^2$. It is often expressed in the form $f(t) = (x(t), y(t))$; and one frequently uses the fact that f is a continuous function of t if both x and y are. Similarly, a *vector field* in the plane

$$\begin{aligned}\mathbf{v}(x, y) &= P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \\ &= (P(x, y), Q(x, y))\end{aligned}$$

is said to be continuous if both P and Q are continuous functions, or equivalently, if $\mathbf{v}$ is continuous as a map of $\mathbb{R}^2$ into $\mathbb{R}^2$. Both of these statements are simply special cases of the preceding theorem.

One way of forming continuous functions that is used a great deal in analysis is to take sums, differences, products, or quotients of continuous real-valued functions. It is a standard theorem that if $f, g : X \to \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous, and f/g is continuous if $g(x) \neq 0$ for all x. We shall consider this theorem in §21.

Yet another method for constructing continuous functions that is familiar from analysis is to take the limit of an infinite sequence of functions. There is a theorem to the effect that if a sequence of continuous real-valued functions of a real variable converges uniformly to a limit function, then the limit function is necessarily continuous. This theorem is called the *Uniform Limit Theorem*. It is used, for instance, to demonstrate the continuity of the trigonometric functions, when one defines these functions rigorously using the infinite series definitions of the sine and cosine. This theorem generalizes to a theorem about maps of an arbitrary topological space X into a metric space Y. We shall prove it in §21.

Exercises

1. Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

3. Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.
(a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.
(b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

7. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.
(b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.
(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous [*Hint:* Use the pasting lemma.]

9. Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|A_\alpha$, is continuous for each α.

(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

(c) An indexed family of sets $\{A_\alpha\}$ is said to be ***locally finite*** if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

11. Let $F : X \times Y \to Z$. We say that F is ***continuous in each variable separately*** if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

12. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0. \\ 0 & \text{if } x \times y = 0 \times 0 \end{cases}$$

(a) Show that F is continuous in each variable separately.

(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

(c) Show that F is not continuous

13. Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f

§19 The Product Topology

We now return, for the remainder of the chapter, to the consideration of various methods for imposing topologies on sets

Previously, we defined a topology on the product $X \times Y$ of two topological spaces. In the present section, we generalize this definition to more general cartesian products.

So let us consider the cartesian products

$$X_1 \times \cdots \times X_n \quad \text{and} \quad X_1 \times X_2 \times \cdots,$$

where each X_i is a topological space There are two possible ways to proceed. One way is to take as basis all sets of the form $U_1 \times \cdots \times U_n$ in the first case, and of the form $U_1 \times U_2 \times \cdots$ in the second case, where U_i is an open set of X_i for each i. This procedure does indeed define a topology on the cartesian product; we shall call it the *box topology*.

Another way to proceed is to generalize the subbasis formulation of the definition, given in §15. In this case, we take as a subbasis all sets of the form $\pi_i^{-1}(U_i)$, where i is any index and U_i is an open set of X_i. We shall call this topology the *product topology*.

How do these topologies differ? Consider the typical basis element B for the second topology. It is a finite intersection of subbasis elements $\pi_i^{-1}(U_i)$, say for $i = i_1, \ldots, i_k$. Then a point $\mathbf{x}$ belongs to B if and only if $\pi_i(\mathbf{x})$ belongs to U_i for $i = i_1, \ldots, i_k$; there is no restriction on $\pi_i(x)$ for other values of i.

It follows that these two topologies agree for the finite cartesian product and differ for the infinite product. What is not clear is why we seem to prefer the second topology. This is the question we shall explore in this section

Before proceeding, however, we shall introduce a more general notion of cartesian product. So far, we have defined the cartesian product of an indexed family of sets only in the cases where the index set was the set $\{1, \ldots, n\}$ or the set $\mathbb{Z}_+$ Now we consider the case where the index set is completely arbitrary.

Definition. Let J be an index set. Given a set X, we define a ***J*-*tuple*** of elements of X to be a function $\mathbf{x} : J \to X$. If α is an element of J, we often denote the value of $\mathbf{x}$ at α by x_α rather than $\mathbf{x}(\alpha)$; we call it the αth ***coordinate*** of $\mathbf{x}$. And we often denote the function $\mathbf{x}$ itself by the symbol

$$(x_\alpha)_{\alpha \in J},$$

which is as close as we can come to a "tuple notation" for an arbitrary index set J. We denote the set of all J-tuples of elements of X by X^J.

Definition. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. The ***cartesian product*** of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha,$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$ That is, it is the set of all functions

$$\mathbf{x} : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Occasionally we denote the product simply by $\prod A_\alpha$, and its general element by (x_α), if the index set is understood

If all the sets A_α are equal to one set X, then the cartesian product $\prod_{\alpha \in J} A_\alpha$ is just the set X^J of *all* J-tuples of elements of X. We sometimes use "tuple notation" for the elements of X^J, and sometimes we use functional notation, depending on which is more convenient.

Definition. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha,$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the ***box topology***

This collection satisfies the first condition for a basis because $\prod X_\alpha$ is itself a basis element; and it satisfies the second condition because the intersection of any two basis elements is another basis element:

$$(\prod_{\alpha \in J} U_\alpha) \cap (\prod_{\alpha \in J} V_\alpha) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha).$$

Now we generalize the subbasis formulation of the definition. Let

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be the function assigning to each element of the product space its βth coordinate,

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta;$$

it is called the ***projection mapping*** associated with the index β.

Definition. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\},$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta.$$

The topology generated by the subbasis $\mathcal{S}$ is called the ***product topology***. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a ***product space***.

To compare these topologies, we consider the basis $\mathcal{B}$ that $\mathcal{S}$ generates. The collection $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$. If we intersect elements belonging to the same one of the sets $\mathcal{S}_\beta$, we do not get anything new, because

$$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta);$$

the intersection of two elements of $\mathcal{S}_\beta$, or of finitely many such elements, is again an element of $\mathcal{S}_\beta$. We get something new only when we intersect elements from different sets $\mathcal{S}_\beta$. The typical element of the basis $\mathcal{B}$ can thus be described as follows: Let $\beta_1, \dots, \beta_n$ be a finite set of distinct indices from the index set J, and let U_{β_i} be an open set in X_{β_i} for $i = 1, \dots, n$. Then

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

is the typical element of $\mathcal{B}$.

Now a point $\mathbf{x} = (x_\alpha)$ is in B if and only if its β_1th coordinate is in U_{β_1}, its β_2th coordinate is in U_{β_2}, and so on. There is no restriction whatever on the αth coordinate of $\mathbf{x}$ if α is not one of the indices $\beta_1, \dots, \beta_n$. As a result, we can write B as the product

$$B = \prod_{\alpha \in J} U_\alpha,$$

where U_α denotes the entire space X_α if $\alpha \neq \beta_1, \dots, \beta_n$.

All this is summarized in the following theorem:

Theorem 19.1 (Comparison of the box and product topologies). *The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.*

Two things are immediately clear. First, for finite products $\prod_{\alpha=1}^n X_\alpha$ the two topologies are precisely the same. Second, the box topology is in general finer than the product topology.

What is not so clear is why we prefer the product topology to the box topology. The answer will appear as we continue our study of topology. We shall find that a number of important theorems about finite products will also hold for arbitrary products if we use the product topology, but not if we use the box topology. As a result, the product topology is extremely important in mathematics. The box topology is not so important; we shall use it primarily for constructing counterexamples. Therefore, we make the following convention:

> *Whenever we consider the product $\prod X_\alpha$, we shall assume it is given the product topology unless we specifically state otherwise.*

Some of the theorems we proved for the product $X \times Y$ hold for the product $\prod X_\alpha$ no matter which topology we use. We list them here; most of the proofs are left to the exercises.

Theorem 19.2. *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

EXAMPLE 1. Consider euclidean n-space $\mathbb{R}^n$. A basis for $\mathbb{R}$ consists of all open intervals in $\mathbb{R}$; hence a basis for the topology of $\mathbb{R}^n$ consists of all products of the form

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n).$$

Since $\mathbb{R}^n$ is a finite product, the box and product topologies agree Whenever we consider $\mathbb{R}^n$, we will assume that it is given this topology, unless we specifically state otherwise

Theorem 19.3. *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Theorem 19.4. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

Theorem 19.5. *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then*

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}.$$

Proof. Let $\mathbf{x} = (x_\alpha)$ be a point of $\prod \bar{A}_\alpha$; we show that $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains $\mathbf{x}$. Since $x_\alpha \in \bar{A}_\alpha$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$ for each α. Then $\mathbf{y} = (y_\alpha)$ belongs to both U and $\prod A_\alpha$. Since U is arbitrary, it follows that $\mathbf{x}$ belongs to the closure of $\prod A_\alpha$.

Conversely, suppose $\mathbf{x} = (x_\alpha)$ lies in the closure of $\prod A_\alpha$, in either topology. We show that for any given index β, we have $x_\beta \in \bar{A}_\beta$. Let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, it contains a point $\mathbf{y} = (y_\alpha)$ of $\prod A_\alpha$. Then y_β belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \in \bar{A}_\beta$. ∎

So far, no reason has appeared for preferring the product to the box topology. It is when we try to generalize our previous theorem about continuity of maps into product spaces that a difference first arises. Here is a theorem that does not hold if $\prod X_\alpha$ is given the box topology:

Theorem 19.6. *Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof. Let π_β be the projection of the product onto its βth factor. The function π_β is continuous, for if U_β is open in X_β, the set $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on X_α. Now suppose that $f : A \to \prod X_\alpha$ is continuous. The function f_β equals the composite $\pi_\beta \circ f$; being the composite of two continuous functions, it is continuous.

Conversely, suppose that each coordinate function f_α is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A, we remarked on this fact when we defined continuous functions. A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$, where β is some index and U_β is open in X_β. Now

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta),$$

because $f_\beta = \pi_\beta \circ f$. Since f_β is continuous, this set is open in A, as desired. ∎

Why does this theorem fail if we use the box topology? Probably the most convincing thing to do is to look at an example.

EXAMPLE 2 Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} X_n,$$

where $X_n = \mathbb{R}$ for each n Let us define a function f $\mathbb{R} \to \mathbb{R}^\omega$ by the equation

$$f(t) = (t, t, t, \ldots),$$

the nth coordinate function of f is the function $f_n(t) = t$. Each of the coordinate functions f_n . $\mathbb{R} \to \mathbb{R}$ is continuous; therefore, the function f is continuous if $\mathbb{R}^\omega$ is given the product topology. But f is not continuous if $\mathbb{R}^\omega$ is given the box topology Consider, for example, the basis element

$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$$

for the box topology. We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. This would mean that $f((-\delta, \delta)) \subset B$, so that, applying π_n to both sides of the inclusion,

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset (-1/n, 1/n)$$

for all n, a contradiction

Exercises

1. Prove Theorem 19.2
2. Prove Theorem 19.3.
3. Prove Theorem 19.4
4. Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.
5. One of the implications stated in Theorem 19.6 holds for the box topology. Which one?
6. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?
7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.
8. Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation
$$h((x_1, x_2, \ldots)) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots).$$
Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?
9. Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq 0$, the cartesian product
$$\prod_{\alpha \in J} A_\alpha$$
is not empty.
10. Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.
    (a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.
    (b) Let
    $$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$
    and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$
    (c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.
    (d) Let $f : A \to \prod X_\alpha$ be defined by the equation
    $$f(a) = (f_\alpha(a))_{\alpha \in J};$$
    let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

§20 The Metric Topology

One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on the set. Topologies given in this way lie at the heart of modern analysis, for example. In this section, we shall define the metric topology and shall give a number of examples. In the next section, we shall consider some of the properties that metric topologies satisfy.

Definition. A ***metric*** on a set X is a function

$$d : X \times X \longrightarrow R$$

having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the ***distance*** between x and y in the metric d Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the ***ϵ-ball centered at x***. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Definition. If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the ***metric topology*** induced by d.

The first condition for a basis is trivial, since $x \in B(x, \epsilon)$ for any $\epsilon > 0$. Before checking the second condition for a basis, we show that if y is a point of the basis element $B(x, \epsilon)$, then there is a basis element $B(y, \delta)$ *centered* at y that is contained in $B(x, \epsilon)$. Define δ to be the positive number $\epsilon - d(x, y)$. Then $B(y, \delta) \subset B(x, \epsilon)$, for if $z \in B(y, \delta)$, then $d(y, z) < \epsilon - d(x, y)$, from which we conclude that

$$d(x, z) \leq d(x, y) + d(y, z) < \epsilon.$$

See Figure 20.1.

Now to check the second condition for a basis, let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$. We have just shown that we can choose positive numbers δ_1 and δ_2 so that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. Letting δ be the smaller of δ_1 and δ_2, we conclude that $B(y, \delta) \subset B_1 \cap B_2$.

Using what we have just proved, we can rephrase the definition of the metric topology as follows:

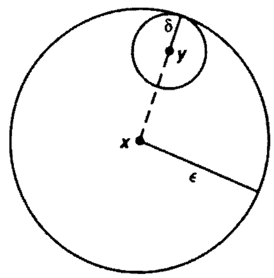

Figure 20.1

> *A set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.*

Clearly this condition implies that U is open. Conversely, if U is open, it contains a basis element $B = B_d(x, \epsilon)$ containing y, and B in turn contains a basis element $B_d(y, \delta)$ centered at y

EXAMPLE 1 Given a set X, define

$$d(x, y) = 1 \quad \text{if } x \neq y,$$
$$d(x, y) = 0 \quad \text{if } x = y$$

It is trivial to check that d is a metric. The topology it induces is the discrete topology; the basis element $B(x, 1)$, for example, consists of the point x alone.

EXAMPLE 2. The standard metric on the real numbers $\mathbb{R}$ is defined by the equation

$$d(x, y) = |x - y|$$

It is easy to check that d is a metric. The topology it induces is the same as the order topology: Each basis element (a, b) for the order topology is a basis element for the metric topology, indeed,

$$(a, b) = B(x, \epsilon),$$

where $x = (a + b)/2$ and $\epsilon = (b - a)/2$. And conversely, each ϵ-ball $B(x, \epsilon)$ equals an open interval the interval $(x - \epsilon, x + \epsilon)$.

Definition. If X is a topological space, X is said to be ***metrizable*** if there exists a metric d on the set X that induces the topology of X. A ***metric space*** is a metrizable space X together with a specific metric d that gives the topology of X.

Many of the spaces important for mathematics are metrizable, but some are not. Metrizability is always a highly desirable attribute for a space to possess, for the existence of a metric gives one a valuable tool for proving theorems about the space.

It is, therefore, a problem of fundamental importance in topology to find conditions on a topological space that will guarantee it is metrizable. One of our goals in Chapter 4 will be to find such conditions; they are expressed there in the famous theorem called *Urysohn's metrization theorem.* Further metrization theorems appear in Chapter 6. In the present section we shall content ourselves with proving merely that $\mathbb{R}^n$ and $\mathbb{R}^\omega$ are metrizable.

Although the metrizability problem is an important problem in topology, the study of metric spaces as such does not properly belong to topology as much as it does to analysis. Metrizability of a space depends only on the topology of the space in question, but properties that involve a specific metric for X in general do not. For instance, one can make the following definition in a metric space.

Definition. Let X be a metric space with metric d. A subset A of X is said to be ***bounded*** if there is some number M such that

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A. If A is bounded and nonempty, the ***diameter*** of A is defined to be the number

$$\operatorname{diam} A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Boundedness of a set is not a topological property, for it depends on the particular metric d that is used for X. For instance, if X is a metric space with metric d, then there exists a metric $\bar{d}$ that gives the topology of X, relative to which *every* subset of X is bounded. It is defined as follows:

Theorem 20.1. *Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation*

$$\bar{d}(x, y) = \min\{d(x, y), 1\}$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the ***standard bounded metric*** corresponding to d.

Proof. Checking the first two conditions for a metric is trivial. Let us check the triangle inequality:

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z).$$

Now if either $d(x, y) \geq 1$ or $d(y, z) \geq 1$, then the right side of this inequality is at least 1, since the left side is (by definition) at most 1, the inequality holds. It remains to consider the case in which $d(x, y) < 1$ and $d(y, z) < 1$. In this case, we have

$$d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z).$$

Since $\bar{d}(x, z) \leq d(x, z)$ by definition, the triangle inequality holds for $\bar{d}$.

Now we note that in any metric space, the collection of ϵ-balls with $\epsilon < 1$ forms a basis for the metric topology, for every basis element containing x contains such an ϵ-ball centered at x. It follows that d and $\bar{d}$ induce the same topology on X, because the collections of ϵ-balls with $\epsilon < 1$ under these two metrics are the same collection. ∎

Now we consider some familiar spaces and show they are metrizable.

Definition. Given $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, we define the ***norm*** of $\mathbf{x}$ by the equation

$$\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2};$$

and we define the ***euclidean metric*** d on $\mathbb{R}^n$ by the equation

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{1/2}.$$

We define the ***square metric*** ρ by the equation

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}.$$

The proof that d is a metric requires some work; it is probably already familiar to you. If not, a proof is outlined in the exercises. We shall seldom have occasion to use this metric on $\mathbb{R}^n$.

To show that ρ is a metric is easier. Only the triangle inequality is nontrivial. From the triangle inequality for $\mathbb{R}$ it follows that for each positive integer i,

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$

Then by definition of ρ,

$$|x_i - z_i| \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}).$$

As a result

$$\rho(\mathbf{x}, \mathbf{z}) = \max\{|x_i - z_i|\} \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}),$$

as desired.

On the real line $\mathbb{R} = \mathbb{R}^1$, these two metrics coincide with the standard metric for $\mathbb{R}$. In the plane $\mathbb{R}^2$, the basis elements under d can be pictured as circular regions, while the basis elements under ρ can be pictured as square regions.

We now show that each of these metrics induces the usual topology on $\mathbb{R}^n$. We need the following lemma:

Lemma 20.2. *Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each x in X and each $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon)$$

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$ Given the basis element $B_d(x, \epsilon)$ for $\mathcal{T}$, there is by Lemma 13.3 a basis element B' for the topology $\mathcal{T}'$ such that $x \in B' \subset B_d(x, \epsilon)$. Within B' we can find a ball $B_{d'}(x, \delta)$ centered at x.

Conversely, suppose the δ-ϵ condition holds Given a basis element B for $\mathcal{T}$ containing x, we can find within B a ball $B_d(x, \epsilon)$ centered at x. By the given condition, there is a δ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$. Then Lemma 13.3 applies to show $\mathcal{T}'$ is finer than $\mathcal{T}$. ■

Theorem 20.3. *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.*

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two points of $\mathbb{R}^n$. It is simple algebra to check that

$$\rho(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{n}\rho(\mathbf{x}, \mathbf{y})$$

The first inequality shows that

$$B_d(\mathbf{x}, \epsilon) \subset B_\rho(\mathbf{x}, \epsilon)$$

for all $\mathbf{x}$ and ϵ, since if $d(\mathbf{x}, \mathbf{y}) < \epsilon$, then $\rho(\mathbf{x}, \mathbf{y}) < \epsilon$ also. Similarly, the second inequality shows that

$$B_\rho(\mathbf{x}, \epsilon/\sqrt{n}) \subset B_d(\mathbf{x}, \epsilon)$$

for all $\mathbf{x}$ and ϵ. It follows from the preceding lemma that the two metric topologies are the same.

Now we show that the product topology is the same as that given by the metric ρ. First, let

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

be a basis element for the product topology, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be an element of B. For each i, there is an ϵ_i such that

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i),$$

choose $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $B_\rho(\mathbf{x}, \epsilon) \subset B$, as you can readily check. As a result, the ρ-topology is finer than the product topology.

Conversely, let $B_\rho(\mathbf{x}, \epsilon)$ be a basis element for the ρ-topology. Given the element $\mathbf{y} \in B_\rho(\mathbf{x}, \epsilon)$, we need to find a basis element B for the product topology such that

$$\mathbf{y} \in B \subset B_\rho(\mathbf{x}, \epsilon).$$

But this is trivial, for

$$B_\rho(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$$

is itself a basis element for the product topology. ■

Now we consider the infinite cartesian product $\mathbb{R}^\omega$. It is natural to try to generalize the metrics d and ρ to this space. For instance, one can attempt to define a metric d on $\mathbb{R}^\omega$ by the equation

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2}.$$

But this equation does not always make sense, for the series in question need not converge. (This equation does define a metric on a certain important subset of $\mathbb{R}^\omega$, however; see the exercises.)

Similarly, one can attempt to generalize the square metric ρ to $\mathbb{R}^\omega$ by defining

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{|x_n - y_n|\}.$$

Again, this formula does not always make sense. If however we replace the usual metric $d(x, y) = |x - y|$ on $\mathbb{R}$ by its bounded counterpart $\bar{d}(x, y) = \min\{|x - y|, 1\}$, then this definition *does* make sense; it gives a metric on $\mathbb{R}^\omega$ called the *uniform metric*.

The uniform metric can be defined more generally on the cartesian product $\mathbb{R}^J$ for arbitrary J, as follows:

Definition. Given an index set J, and given points $\mathbf{x} = (x_\alpha)_{\alpha\in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha\in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. It is easy to check that $\bar{\rho}$ is indeed a metric; it is called the ***uniform metric*** on $\mathbb{R}^J$, and the topology it induces is called the ***uniform topology***.

The relation between this topology and the product and box topologies is the following:

Theorem 20.4. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.*

Proof. Suppose that we are given a point $\mathbf{x} = (x_\alpha)_{\alpha\in J}$ and a product topology basis element $\prod U_\alpha$ about $\mathbf{x}$. Let $\alpha_1, \dots, \alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$. Then for each i, choose $\epsilon_i > 0$ so that the ϵ_i-ball centered at x_{α_i} in the $\bar{d}$ metric is contained in U_{α_i}; this we can do because U_{α_i} is open in $\mathbb{R}$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$; then the ϵ-ball centered at $\mathbf{x}$ in the $\bar{\rho}$ metric is contained in $\prod U_\alpha$. For if $\mathbf{z}$ is a point of $\mathbb{R}^J$ such that $\bar{\rho}(\mathbf{x}, \mathbf{z}) < \epsilon$, then $\bar{d}(x_\alpha, z_\alpha) < \epsilon$ for all α, so that $\mathbf{z} \in \prod U_\alpha$. It follows that the uniform topology is finer than the product topology.

On the other hand, let B be the ϵ-ball centered at $\mathbf{x}$ in the $\bar{\rho}$ metric. Then the box neighborhood

$$U = \prod(x_\alpha - \tfrac{1}{2}\epsilon, x_\alpha + \tfrac{1}{2}\epsilon)$$

of $\mathbf{x}$ is contained in B. For if $\mathbf{y} \in U$, then $\bar{d}(x_\alpha, y_\alpha) < \frac{1}{2}\epsilon$ for all α, so that $\bar{\rho}(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2}\epsilon$.

Showing these three topologies are different if J is infinite is a task we leave to the exercises. ■

In the case where J is infinite, we still have not determined whether $\mathbb{R}^J$ is metrizable in either the box or the product topology. It turns out that the only one of these cases where $\mathbb{R}^J$ *is* metrizable is the case where J is countable and $\mathbb{R}^J$ has the product topology. As we shall see.

Theorem 20.5. *Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x}$ and $\mathbf{y}$ are two points of $\mathbb{R}^\omega$, define*

$$D(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof. The properties of a metric are satisfied trivially except for the triangle inequality, which is proved by noting that for all i,

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}),$$

so that

$$\sup\left\{\frac{\bar{d}(x_i, z_i)}{i}\right\} \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}).$$

The fact that D gives the product topology requires a little more work. First, let U be open in the metric topology and let $\mathbf{x} \in U$; we find an open set V in the product topology such that $\mathbf{x} \in V \subset U$. Choose an ϵ-ball $B_D(\mathbf{x}, \epsilon)$ lying in U. Then choose N large enough that $1/N < \epsilon$. Finally, let V be the basis element for the product topology

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots.$$

We assert that $V \subset B_D(\mathbf{x}, \epsilon)$: Given any $\mathbf{y}$ in $\mathbb{R}^\omega$,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N} \quad \text{for } i \geq N.$$

Therefore,

$$D(\mathbf{x}, \mathbf{y}) \leq \max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \ldots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}.$$

If $\mathbf{y}$ is in V, this expression is less than ϵ, so that $V \subset B_D(\mathbf{x}, \epsilon)$, as desired.

Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \dots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i. Given $\mathbf{x} \in U$, we find an open set V of the metric topology such that $\mathbf{x} \in V \subset U$. Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centered about x_i and lying in U_i for $i = \alpha_1, \dots, \alpha_n$; choose each $\epsilon_i \leq 1$. Then define

$$\epsilon = \min\{\epsilon_i / i \mid i = \alpha_1, \dots, \alpha_n\}.$$

We assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U.$$

Let $\mathbf{y}$ be a point of $B_D(\mathbf{x}, \epsilon)$. Then for all i,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon.$$

Now if $i = \alpha_1, \dots, \alpha_n$, then $\epsilon \leq \epsilon_i / i$, so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$; it follows that $|x_i - y_i| < \epsilon_i$. Therefore, $\mathbf{y} \in \prod U_i$, as desired. ∎

Exercises

1. (a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p \right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

3. Let X be a metric space with metric d.
   (a) Show that $d : X \times X \to \mathbb{R}$ is continuous.
   (b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

4. Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$.

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$f(t) = (t, 2t, 3t, \dots),$$
$$g(t) = (t, t, t, \dots),$$
$$h(t) = (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \dots).$$

(b) In which topologies do the following sequences converge?

$$\begin{aligned} \mathbf{w}_1 &= (1, 1, 1, 1, \dots), & \mathbf{x}_1 &= (1, 1, 1, 1, \dots), \\ \mathbf{w}_2 &= (0, 2, 2, 2, \dots), & \mathbf{x}_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots), \\ \mathbf{w}_3 &= (0, 0, 3, 3, \dots), & \mathbf{x}_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3} \dots), \\ &\dots & &\dots \\ \mathbf{y}_1 &= (1, 0, 0, 0, \dots), & \mathbf{z}_1 &= (1, 1, 0, 0, \dots), \\ \mathbf{y}_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \dots), & \mathbf{z}_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \dots), \\ \mathbf{y}_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \dots), & \mathbf{z}_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \dots), \\ &\dots & &\dots \end{aligned}$$

5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

6. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots.$$

(a) Show that $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ-ball $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.

(b) Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.

(c) Show that

$$B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

7. Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined in Exercise 8 of §19; give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i and b_i is h continuous? a homeomorphism?

8. Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2}$$

defines a metric on X. (See Exercise 10.) On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We have also the topology given by the metric d, which we call the *ℓ^2-topology*. (Read "little ell two.")

(a) Show that on X, we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}.$$

(b) The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

(c) The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

is contained in X; it is called the ***Hilbert cube***. Compare the four topologies that H inherits as a subspace of X.

9. Show that the euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n),$$
$$c\mathbf{x} = (cx_1, \dots, cx_n),$$
$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

(a) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$.

(b) Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$. [*Hint:* If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

(c) Show that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [*Hint:* Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply (b).]

(d) Verify that d is a metric.

10. Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \dots)$ such that $\sum x_i^2$ converges. (You may assume the standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i y_i|$ converges. [*Hint:* Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let $c \in \mathbb{R}$. Show that if $\mathbf{x}, \mathbf{y} \in X$, then so are $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$.

(c) Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

is a well-defined metric on X.