Study Material

Dept. of Applied Mathematics with Oceanology and Computer Programming

Paper No. - MTM 205

Paper Name - Continuum Mechanics

Semester - 2

Topic of Lecture - Complex Potential, its properties, Source, Sink, Doublet

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Lecture No - 06.

6.17 Complex Potential:

At this stage we introduce the notion of a function of a complex variable. Suppose φ and ψ represent velocity potential and stream function of a two dimensional irrotational motion of a perfect fluid. Let

$$w = f(z) = \varphi(x, y) + i\psi(x, y) \text{ where } z = x + iy, i = \sqrt{-1}.$$

Then w is defined as complex potential of the fluid.

Property 1. w is an analytic function.

Proof. Since $w = f(z) = \varphi(x, y) + i\psi(x, y)$ where $z = x + iy$

So, we must have

$$u = -\frac{\partial\varphi}{\partial x},\ v = -\frac{\partial\varphi}{\partial y}$$

and $$u = -\frac{\partial\psi}{\partial y},\ v = \frac{\partial\psi}{\partial x}$$

where u and v are velocity components of the fluid.

$$\therefore \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \text{ and } \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

which are Cauchy-Riemann equations. Thus Cauchy-Riemann equation are satisfied so that w is analytic function of z.

Conversely if w is analytic function, then its real and imaginary i.e., φ and ψ gives the velocity potential and stream function for a possible two dimensional irrotational fluid motion.

Property 2. $\left|\frac{dw}{dz}\right|$ is the magnitude of the velocity of the fluid at any point.

Proof. Since $w = f(z) = \varphi(x, y) + i\psi(x, y), z = x + iy$

$$\therefore \frac{dw}{dz} = f'(z) = \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{Lt} \left[\frac{\delta\varphi + i\delta\psi}{\delta x + i\delta y}\right]$$

where $\delta\varphi = \varphi(x+\delta x, y+\delta y) - \varphi(x,y)$

$\delta\psi = \psi(x+\delta x, y+\delta y) - \psi(x,y)$

At first we keep y as a constant, then

$$\frac{dw}{dz} = \underset{\delta x \to 0}{Lt}\left[\frac{\delta\varphi + i\delta\psi}{\delta x}\right] = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = -u + iv$$

where $u = -\frac{\partial\varphi}{\partial x} = -\frac{\partial\psi}{\partial y}, v = -\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial x}$.

Similarly, if we keep x as a constant then

$$\frac{dw}{dz} = \underset{\delta y \to 0}{Lt}\left[\frac{\delta\varphi + i\delta\psi}{i\delta y}\right] = -i\frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial y} = -u + iv$$

Hence, $\left|\frac{dw}{dz}\right| = \sqrt{u^2 + v^2}$ = magnitude of the velocity.

Therefore $\left|\frac{dw}{dz}\right|$ represents the magnitude of the velocity.

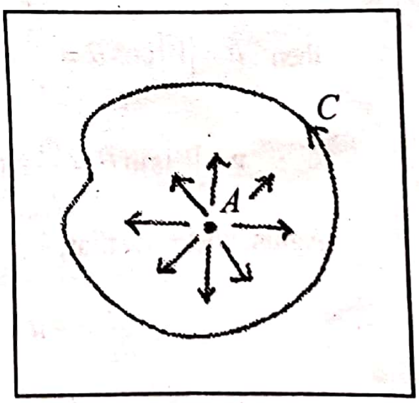
Stagnation Points: The points, where the fluid velocity is zero, are called stagnation points.

Thus for stagnation points, we must have

$$\frac{dw}{dz} = 0.$$

6.18 Two Dimensional Sources, Sinks:

i) Sources: A source is a point from which liquid is emitted radially and symmetrically in all directions in xy-plane.



ii) Sink: A point to which fluid is flowing in symmetrically and radially in all directions is called sink. This sink is a negative of source.

Source is a point at which liquid is continuously created and sink is a point at which liquid is continuously annihilated. Really speaking, source and sink are purely abstract conceptions which do not occur in nature.

iii) Strength: Strength of a source is defined as total volume of flow per unit time from it.

Thus if $2\pi m$ is the total volume of flow across any small circle surrounding the source, then m is called strength of the source. Sink is a source of strength $(-m)$.

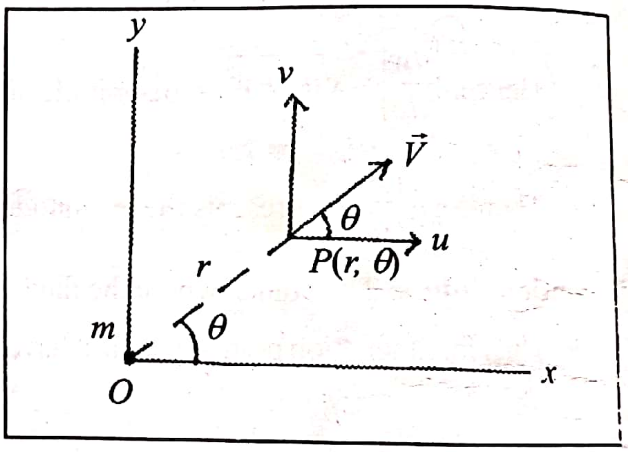
If we consider, for C the circle centre at A of radius r, the speed of flow $|\vec{V}|$ is everywhere of the same on C and the mass is now $2\pi r\rho|\vec{V}|$ per unit length. Thus

$$(2\pi m)\rho = 2\pi r\rho|\vec{V}| \text{ for unit length}$$

$$\therefore |\vec{V}| = \frac{m}{r} \text{ or, } -\frac{\partial\varphi}{\partial r} = \frac{m}{r}.$$

6.19 Complex Potential Due to a Source:

Let us consider a source of strength m at the origin. Let w be the complex potential at $P(r,\theta)$ due to this source. The velocity at P due to the source is purely radial, i.e., along $\overline{OP}$ and let this velocity be $|\vec{V}|$. Now the flux across a circle of radius r surrounding the source at O is $2\pi r|\vec{V}|$. So



$$2\pi r|\vec{V}| = 2\pi m$$

$$\Rightarrow |\vec{V}| = \frac{m}{r}$$

Let u, v be the velocity components along Ox and Oy,

then $u = |\vec{V}|\cos\theta = \frac{m}{r}\cos\theta$

$$v = |\vec{V}|\sin\theta = \frac{m}{r}\sin\theta$$

Again, we know that,

$$\frac{dw}{dz} = -u + iv = \frac{m}{r}[-\cos\theta + i\sin\theta] = -\frac{m}{r}e^{i\theta}$$

$$= -\frac{m}{re^{i\theta}} = -\frac{m}{z}$$

$$\therefore dw = -m\frac{dz}{z}$$

Integrating, $w=-m\log_e z$ (Neglecting constant of integration).

If the source of strength m is at a point $z=z_1$ in place of $z=0$, then by shifting the origin, we have

$$w=-m\log_e(z-z_1).$$

Corollary: Complex potential at a point due to the sources of strengths $m_1, m_2, m_3, \ldots$ situated at $z_1, z_2, z_3, \ldots$

Proof. We have the complex potential w due to a source of strength m situated at $z=z_1$ is

$$w=-m\log_e(z-z_1)$$

Hence the required complex potential due to the sources of strengths $m_1, m_2, m_3, \ldots$ situated at $z_1, z_2, z_3, \ldots$ is

$$w=-m_1\log_e(z-z_1)-m_2\log_e(z-z_2)-m_3\log_e(z-z_3)-\ldots\ldots$$

$$=-\sum_{j=1}^{\infty} m_j\log_e(z-z_j)$$

which $\Rightarrow \varphi=-\sum_{j=1}^{\infty} m_n\log_e r_n$

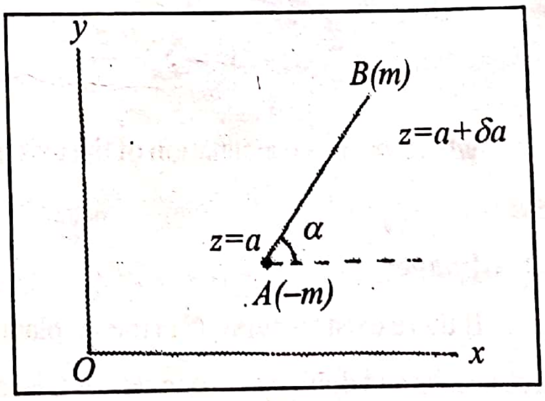
$$\psi=-\sum_{n=1}^{\infty} m_n\theta_n, \text{ where } z-z_n=r_n e^{i\theta_n}.$$

6.20 Doublet:

A doublet is defined as a combination of the equal and opposite sources, i.e., a source and a sink of strengths (+m) and (–m) respectively, situated at a distance δs apart such that the product $m\delta s$ is finite.

Strength of doublet: If $m\delta s=\mu=$ finite, where $m\to\infty$, $\delta s\to 0$ then μ is called strength of the doublet and line δs is called the axis of the doublet and its direction is taken from sink to source.

6.21 Complex Potential for a Doublet:



Let μ be the strength of a doublet AB formed by a sink $-m$ at $A(z=a)$ and a source $+m$ at $B(z=a+\delta a)$. Then

$$\left.\begin{array}{r}\mu=m.AB\\ \text{and}\ \delta a=ABe^{i\alpha}\end{array}\right\}\left(\because z=re^{i\theta}\right) \quad \ldots\ldots(i)$$

where α is the inclination of the axis of the doublet with Ox.

If w be the complex potential due to this doublet at any point $P(z)$ then

$$w = +m\log(z-a) - m\log\left(z-\overline{a+\delta a}\right)$$

$$= -m\log_e\left(\frac{z-a-\delta a}{z-a}\right)$$

$$= -m\log_e\left(1-\frac{\delta a}{z-a}\right)$$

$$= -m\left[-\frac{\delta a}{z-a} - \left(\frac{\delta a}{z-a}\right)\cdot\frac{1}{2} - \ldots\ldots\right]$$

$$= +m\left(\frac{\delta a}{z-a}\right) \text{ (upto first approximation)}$$

$$= \frac{m\, AB.e^{i\alpha}}{z-a}$$

$$= \frac{\mu e^{i\alpha}}{z-a} \text{ [using (i)]}$$

$$\therefore w = \mu\frac{e^{i\alpha}}{z-a}$$

Note-1 If the axis of the doublet is along x-axis, then $\alpha = 0^0$

so that $$w = \frac{\mu}{z-a}.$$

Note-2 If the axis of the doublet is along x-axis and the doublet is at the origin, then $\alpha = 0^0, a = 0$, so that

$$w = \frac{\mu}{z}$$

Note-3 If a system consists of doublets of strengths $\mu_1, \mu_2, \ldots$ placed at $z = a_1, a_2, \ldots$, then

$$w = \sum_{n=1}^{\infty}\frac{\mu_n e^{i\alpha_n}}{z-a_n}$$

where α_n is the inclination of the axis of the doublet of strength μ_n with Ox.