Soft Computing
Paper: MTM 402 (Unit-2)
Fuzzy Logic

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Fuzzy logic

- Fuzzy logic is useful in representing human knowledge in a specific domain of application and reasoning with that knowledge to make useful inferences or actions.
- Fuzzy logic is a form of multi-valued logic derived from fuzzy set theory to deal with reasoning that is approximate rather than precise. Fuzzy logic is not a vague logic system, but a system of logic for dealing with vague concepts.
- Fuzzy logic is a logic; its ultimate goal is to provide foundations for approximate reasoning using imprecise propositions based on fuzzy set theory, in a way similar to the classical reasoning using precise propositions based on the classical set theory.

Fuzzy System

A Fuzzy System can be contrasted with a conventional (crisp) system in three main ways:

- A Linguistic Variable is defined as a variable whose values are sentences in a natural or artificial language. Thus, "if tall", "not tall", "very tall", "very very tall", etc. are values of height, then height is a linguistic variable.
- Fuzzy Conditional Statements are expressions of the form "If $\tilde{A}$ THEN $\tilde{B}$", where $\tilde{A}$ and $\tilde{B}$ have fuzzy meaning, e.g. "If x is small THEN y is large", where small and large are viewed as labels of fuzzy sets.
- A Fuzzy Algorithm is an ordered sequence of instructions which may contain fuzzy assignment and fuzzy conditional statements, e.g., x = very small, IF x is small THEN y is large. The execution of such instructions is governed by the compositional rule of inference.

Fuzzy System

There are two important ideas in fuzzy systems theory:

- The real world is too complicated for precise descriptions to be obtained; therefore, approximation (or fuzziness) must be introduced in order to obtain a reasonable model.
- Now-a-days, human knowledge becomes increasingly important. We need a theory to formulate human knowledge in a systematic manner and put it into engineering systems, together with other information like mathematical models and sensory measurements.

Fuzzy Relations

Let X, Y be universal sets, then

$$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) : (x, y) \in X \times Y\}$$

is called a fuzzy relation in $X \times Y$. A fuzzy relation $\tilde{R}$ is a mapping from the Cartesian space $X \times Y$ to the interval [0, 1], where the strength of the mapping is expressed by the membership function of ordered pairs from the two universes, or $\mu_{\tilde{R}}(x, y)$.
For discrete supports, fuzzy relations can be defined by matrices. For example, let $X = Y$, $A = \{a_1, a_2, a_3, a_4\} = \{1, 2, 3, 4\}$ and $B = \{b_1, b_2, b_3\} = \{0, 1, 2\}$. Then the following matrix expresses a fuzzy relation $\tilde{R}_1$ which is defined as "a is considerably larger than b"

$$\tilde{R}_1 : \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 0.6 & 0.6 & 0 \\ 0.8 & 0.7 & 0 \\ 0.9 & 0.8 & 0.4 \\ 1.0 & 0.9 & 0.5 \end{pmatrix} \end{array}$$

Operations on Fuzzy Relations

Let $\tilde{R}$ and $\tilde{S}$ be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations:

$$Union: \quad \mu_{\tilde{R}\cup\tilde{S}}(x,y) = \max(\mu_{\tilde{R}}(x,y),\, \mu_{\tilde{S}}(x,y)) \qquad (1)$$

$$Intersection: \quad \mu_{\tilde{R}\cap\tilde{S}}(x,y) = \min(\mu_{\tilde{R}}(x,y),\, \mu_{\tilde{S}}(x,y)) \qquad (2)$$

$$Complement: \quad \mu_{\tilde{R}'}(x,y) = 1 - \mu_{\tilde{R}}(x,y) \qquad (3)$$

$$Containment: \quad \tilde{R} \subset \tilde{S} \Rightarrow \mu_{\tilde{R}}(x,y) \le \mu_{\tilde{S}}(x,y) \qquad (4)$$

Let $\tilde{R}_2$ be the fuzzy relation defined as "a is considerably close to b" and expresses by the following matrix:

$$\tilde{R}_2: \begin{matrix} & b_1 & b_2 & b_3 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{matrix} \begin{pmatrix} 0.2 & 0.2 & 0.5 \\ 0.1 & 0.1 & 1.0 \\ 0.0 & 0.0 & 0.3 \\ 0.0 & 0.0 & 0.5 \end{pmatrix}$$

Operations on Fuzzy Relations

Then,

$$\tilde{R}_1 \cup \tilde{R}_2 : \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 0.6 & 0.6 & 0.5 \\ 0.8 & 0.7 & 1.0 \\ 0.9 & 0.8 & 0.4 \\ 1.0 & 0.9 & 0.5 \end{pmatrix} \end{array}$$

and

$$\tilde{R}_1 \cap \tilde{R}_2 : \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.0 \\ 0.1 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.3 \\ 0.0 & 0.0 & 0.5 \end{pmatrix} \end{array}$$

Here, in this example, $\tilde{R}_1 \cup \tilde{R}_2$ means that “a is either considerably larger than or considerably close to b”, and $\tilde{R}_1 \cap \tilde{R}_2$ means that “a is considerably larger than, as well as considerably close to b”.

Max-Min composition

Example 2.1.

Find the relational matrix of the concept "a young tall man", where "Young man"$= \frac{0}{15} + \frac{0.5}{20} + \frac{1}{25} + \frac{0.5}{30} + \frac{0}{35}$ and "Tall man" $= \frac{0}{170} + \frac{0.5}{175} + \frac{1}{180} + \frac{1}{185} + \frac{1}{190}$.

Fuzzy relations in different product spaces can be combined with each other by composition. Among some other important compositions, the max-min composition is the most useful one in applications.

Let $\tilde{R}_1(x, y)$ and $\tilde{R}_2(y, z)$ be two fuzzy relations. Their max-min composition is defined to be the new relation $\tilde{R}(x, z) = \tilde{R}_1(x, y) \circ \tilde{R}_2(y, z)$ with the membership function

$$\mu_{\tilde{R}}(x, z) = \max_{y \in Y}\{\min\{\mu_{\tilde{R}_1}(x, y),\ \mu_{\tilde{R}_2}(y, z)\}\},\ (x,\ z) \in X \times Z$$

Max-Min composition

Example 2.2.

$\tilde{R}_1$ is a relation that describes an interconnection between color and ripeness of a tomato, and $\tilde{R}_2$ represents an interconnection between ripeness and taste of a tomato. Present a fuzzy relational matrices for the MAX-MIN compositions. Where

$$\tilde{R}_1 : \begin{array}{c} \\ Green \\ Yellow \\ Red \end{array} \begin{array}{c} \begin{array}{ccc} Unripe & Semiripe & Ripe \end{array} \\ \begin{pmatrix} 1 & 0.5 & 0 \\ 0.3 & 1 & 0.4 \\ 0 & 0.2 & 1 \end{pmatrix} \end{array}$$

and

$$\tilde{R}_2 : \begin{array}{c} \\ Unripe \\ Semiripe \\ Ripe \end{array} \begin{array}{c} \begin{array}{ccc} Sour & Sweet-sour & Sweet \end{array} \\ \begin{pmatrix} 1 & 0.2 & 0 \\ 0.7 & 1 & 0.3 \\ 0 & 0.7 & 1 \end{pmatrix} \end{array}$$

Classical reasoning

We first recall how the classical reasoning works for precise propositions with two-valued logic. The following syllogism is an example of such reasoning in linguistic terms:

(i). Everyone who is 40 years old or older is old.

(ii). David is 40 years old and Mary is 39 years old.

(iii). David is old but Mary is not.

This is a very precise deductive inference, correct in the sense of the two-valued logic.

In this classical (precise) reasoning using the two-valued logic, when the (output) logical variable represented by a logical formula is always true regardless of the truth values of the (input) logical variables, it is called a **tautology**. If, on the contrary, it is always false, then it is called a **contradiction**. Various tautologies can be used for making deductive inferences, which are referred to as inference rules.

Classical reasoning

The four frequently used inference rules in classical reasoning are:

$$\text{modus ponens:} \quad (p \wedge (p \Rightarrow q)) \Rightarrow q \tag{5}$$

$$\text{modus tollens:} \quad (\bar{q} \wedge (p \Rightarrow q)) \Rightarrow \bar{p} \tag{6}$$

$$\text{syllogism:} \quad (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r) \tag{7}$$

$$\text{contraposition:} \quad (p \Rightarrow q) \Rightarrow (\bar{q} \Rightarrow \bar{p}) \tag{8}$$

These inference rules are very easily understood and, indeed, have been commonly used in one's daily life. For example, the modus ponens is interpreted as follows: *IF "p is true" AND the statement is true" is true THEN "q is true".*

Using this logic, we see that the deductive inference *IF "40 years ol or older is old T* is a modus ponens.

Classical reasoning

The deductive inference

> *IF "Mary is not old" AND "IF Mary is 40 years old or older THEN Mary is old" THEN "Mary is not 40 years old or older"*

is a modus tollens.
With the above discussion in mind, we now consider the following example of approximate reasoning in linguistic terms that cannot be handled by the classical (precise) reasoning using two-valued logic:

(i). Everyone who is 40 to 70 years old is old but is very old if he (she) is 71 years old or above; everyone who is 20 to 39 years old is young but is very young if he (she) is 19 years old or below.

(ii). David is 40 years old and Mary is 39 years old.

(iii). David is old but not very old; Mary is young but not very young.

Classical reasoning

This is of course a meaningful deductive inference, which has been frequently used in one's daily life. This is an example of what is called approximate reasoning.
In order to deal with such imprecise inference, fuzzy logic can be employed. Briefly, fuzzy logic allows the imprecise linguistic terms such as:

- fuzzy predicates: old, rare, severe, expensive, high, fast
- fuzzy quantifiers: many, few, usually, almost, little, much
- fuzzy truth values: very true, true, unlikely true, mostly false, false, definitely false

Fuzzy Proposition

A fuzzy logic proposition, $\tilde{P}$, is a statement involving some concept without clearly defined boundaries. Linguistic statements that tend to express subjective ideas and that can be interpreted slightly different by various individuals typically involve fuzzy propositions. Most natural language is fuzzy, in that it involves vague and imprecise terms. Statements describing a person's height or weight or assessments of people's preferences about colors or menus can be used as examples of fuzzy propositions. The truth value assigned to $\tilde{P}$ can be any value on the interval [0, 1]. The assignment of the truth value to a proposition is actually a mapping from the universe U to the interval [0, 1] of truth values, T, given as

$$T : u \in U \to [0, 1]$$

Fuzzy Proposition and Connectives

We assign a fuzzy proposition to a fuzzy set in the universe of discourse. Suppose fuzzy proposition $\tilde{P}$ is assigned to fuzzy set $\tilde{A}$; then the truth value of a proposition, denoted $T(\tilde{P})$, is given by

$$T(\tilde{P}) = \mu_{\tilde{A}}(x), \text{ where } 0 \le \mu_{\tilde{A}}(x) \le 1 \tag{9}$$

Equation (9) indicates that the degree of truth for the proposition $\tilde{P} : x \in \tilde{A}$ is equal to the membership grade of x in the fuzzy set $\tilde{A}$.
The logical connectives of negation, disjunction, conjunction, and implication are also defined for a fuzzy logic. These connectives for two simple propositions: proposition $\tilde{P}$ defined on fuzzy set $\tilde{A}$ and proposition $\tilde{Q}$ defined on fuzzy set $\tilde{B}$, are given as:

$$\text{Negation:} \quad T(\bar{\tilde{P}}) = 1 - T(\tilde{P}) \tag{10}$$

$$\text{Disjunction:} \quad \tilde{P} \vee \tilde{Q} : x \text{ is } \tilde{A} \text{ or } \tilde{B}, \quad T(\tilde{P} \vee \tilde{Q}) = \max(T(\tilde{P}), T(\tilde{Q})) \tag{11}$$

$$\text{Conjunction:} \quad \tilde{P} \wedge \tilde{Q} : x \text{ is } \tilde{A} \text{ and } \tilde{B}, \quad T(\tilde{P} \wedge \tilde{Q}) = \min(T(\tilde{P}), T(\tilde{Q})) \tag{12}$$

$$\text{Implication:} \quad \tilde{P} \to \tilde{Q} : x \text{ is } \tilde{A}, \text{ then x is } \tilde{B}, \quad T(\tilde{P} \to \tilde{Q}) = T(\bar{\tilde{P}} \vee \tilde{Q}) \tag{13}$$

Fuzzy Rules

In the field of artificial intelligence there are various ways to represent knowledge. The most common way to represent human knowledge is to form it into natural language expressions of the type

IF premise (antecedent), THEN conclusion (consequent)

The form in above expression is commonly referred to as the IF-THEN rule-based form.
The implication connective can be modelled in rule-based form;

$$\tilde{P} \to \tilde{Q} \text{ is, IF } x \text{ is } \tilde{A}, \text{ THEN y is } \tilde{B}$$

and it is equivalent to the following fuzzy relation, $\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times Y)$.
The membership function of $\tilde{R}$ is expressed by the following formula:

$$\mu_{\tilde{R}}(x, y) = \max[(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)), (1 - \mu_{\tilde{A}}(x))] \quad (14)$$

When the logical conditional implication is of the compound form;

$$\text{IF } x \text{ is } \tilde{A}, \text{ THEN y is } \tilde{B}, \text{ ELSE y is } \tilde{C}$$

and it is equivalent to the following fuzzy relation, $\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{C})$.
The membership function of $\tilde{R}$ is expressed by the following formula:

$$\mu_{\tilde{R}}(x, y) = \max[(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)), ((1 - \mu_{\tilde{A}}(x)) \wedge \mu_{\tilde{C}}(y))] \quad (15)$$

Fuzzy Rules

Table 1: The canonical form for a fuzzy rule-based system

Rule 1 :	IF condition C^1 , THEN restriction R^1
Rule 2:	IF condition C^2, THEN restriction R^2
.	
.	
.	
Rule r:	IF condition C^r, THEN restriction R^r

By using the basic properties and operations defined for fuzzy sets, any compound rule structure may be decomposed and reduced to a number of simple canonical rules as given in Table 1. The fuzzy level of understanding and describing a complex system is expressed in the form of a set of restrictions on the output based on certain conditions of the input. These restriction statements are usually connected by linguistic connectives such as “AND”, “OR”, or “ELSE”. The most common structures for decomposition of fuzzy rules are follows:

Fuzzy Rules

Multiple conjunctive antecedents: Let the fuzzy rule of the form:

$$\text{IF x is } \tilde{A}^1 \text{ AND } \tilde{A}^2 \text{ AND } \ldots \text{ AND } \tilde{A}^L \text{ THEN y is } \tilde{B}^s$$

Assuming a new fuzzy subset $\tilde{A}^s$ as

$$\tilde{A}^s = \tilde{A}^1 \cap \tilde{A}^2 \cap \ldots \cap \tilde{A}^L$$

expressed by means of membership function

$$\mu_{\tilde{A}^s}(x) = \min[\mu_{\tilde{A}^1}(x), \mu_{\tilde{A}^2}(x), \ldots \mu_{\tilde{A}^L}(x)]$$

based on the definition of the standard fuzzy intersection operation, the compound rule may be rewritten as

$$\text{IF x is } \tilde{A}^s \text{ THEN y is } \tilde{B}^s$$

Multiple disjunctive antecedents: Let the fuzzy rule of the form:

$$\text{IF x is } \tilde{A}^1 \text{ OR } \tilde{A}^2 \text{ OR } \ldots \text{ OR } \tilde{A}^L \text{ THEN y is } \tilde{B}^s$$

could be rewritten as IF x is $\tilde{A}^s$ THEN y is $\tilde{B}^s$, where the fuzzy set $\tilde{A}^s$ is defined as

$$\tilde{A}^s = \tilde{A}^1 \cup \tilde{A}^2 \cup \ldots \cup \tilde{A}^L$$

$$\mu_{\tilde{A}^s}(x) = \max[\mu_{\tilde{A}^1}(x), \mu_{\tilde{A}^2}(x), \ldots \mu_{\tilde{A}^L}(x)]$$

which is based on the definition of the standard fuzzy union operation.

Fuzzy Rules

Aggregation of fuzzy rules:
Most rule-based systems involve more than one rule. The process of obtaining the overall consequent (conclusion) from the individual consequents contributed by each rule in the rule-base is known as aggregation of rules. In determining an aggregation strategy, two simple extreme cases exist

1. **Conjunctive system of rules:** In the case of a system of rules that must be jointly satisfied, the rules are connected by "AND" connectives. In this case the aggregated output (consequent), $\tilde{B}$, is found by the fuzzy intersection of all individual rule consequents, $\tilde{B}^i$, where $i = 1, 2, \ldots r$ as

$$\tilde{B} = \tilde{B}^1 \, AND \, \tilde{B}^2 \, AND \ldots AND \, \tilde{B}^r$$

or, $\tilde{B} = \tilde{B}^1 \cap \tilde{B}^2 \cap \ldots \cap \tilde{B}^r$, which is defined by the membership function

$$\mu_{\tilde{B}}(y) = \min[\mu_{\tilde{B}^1}(y), \mu_{\tilde{B}^2}(y), \ldots, \mu_{\tilde{B}^r}(y)], \text{ for } y \in Y$$

Fuzzy Rules

2. **Disjunctive system of rules:** For the case of a disjunctive system of rules where the satisfaction of at least one rule is required, the rules are connected by the "OR" connectives. In this case the aggregated output is found by the fuzzy union of all individual rule contributions, as

$$\tilde{B} = \tilde{B}^1 \, OR \, \tilde{B}^2 \, OR \ldots OR \, \tilde{B}^r$$

or, $\tilde{B} = \tilde{B}^1 \cup \tilde{B}^2 \cup \ldots \cup \tilde{B}^r$ which is defined by the membership function

$$\mu_{\tilde{B}}(y) = \max[\mu_{\tilde{B}^1}(y), \mu_{\tilde{B}^2}(y), \ldots, \mu_{\tilde{B}^r}(y)], \text{ for } y \in Y$$

Fuzzy Inference

The aim of fuzzy logic is to provide foundations for approximate reasoning with imprecise propositions using fuzzy set theory as the principal tool. In order to achieve this goal, the generalized modus ponens, generalized modus tollens, and generalized hypothetical syllogism were proposed, which are the fundamental principles in fuzzy logic. These generalization are based on compositional rule of inference.

Generalized Modus Ponens: The modus ponens of the classical logic cannot be used in the fuzzy logic environment because such an inference can take place if, and only if, the fact or premise is exactly the same as the antecedent of the IF-THEN rule. In fuzzy logic the generalized modus ponens is used. It allows an inference when the antecedent is only partly known or when the fact is only similar but not equal to it. The modus ponens is generalized to get fuzzy inference rules as follows

Implication:	IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Fact:	x is $\tilde{A}'$
Conclusion:	y is $\tilde{B}'$

For fuzzy sets $\tilde{A}$, $\tilde{A}'$, $\tilde{B}$ and $\tilde{B}'$ and given fuzzy propositions (fact or premise and implication), we infer a new fuzzy proposition for the conclusion such that closer the set $\tilde{A}'$ to $\tilde{A}$, the closer the set $\tilde{B}'$ and $\tilde{B}$

Fuzzy Inference

To compute the membership function of $\tilde{B}'$, the max-min composition of fuzzy set $\tilde{A}'$ with the fuzzy relation $\tilde{R}$, which is known implication relation (IF-THEN rule) is used i.e.

$$\tilde{B}' = \tilde{A}' \circ \tilde{R}$$

In terms of membership function,

$$\mu_{\tilde{B}'}(y) = \max_{x \in X}[\min[\mu_{\tilde{A}'}(x),\, \mu_{\tilde{R}}(x,\, y)]]$$

A typical problem in this fuzzy approximate reasoning is as follows:

Implication:	"a red apple is a ripe apple"
Fact:	"the apple is very red"
Conclusion:	"the apple is very ripe"

Fuzzy Inference

Generalized Modus Tollens: The classical Modus Tollens can be framed for fuzzy inference rules as follows:

Implication:	IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Fact:	y is $\tilde{B}'$
Conclusion:	x is $\tilde{A}'$

For fuzzy sets $\tilde{A}$, $\tilde{A}'$, $\tilde{B}$ and $\tilde{B}'$ and given fuzzy propositions (fact or premise and implication), we infer a new fuzzy proposition for the conclusion such that closer the set $\tilde{A}'$ to $\tilde{A}$, the closer the set $\tilde{B}'$ and $\tilde{B}$.

Fuzzy Inference

To compute the membership function of $\tilde{A}'$, the max-min composition of fuzzy set $\tilde{B}'$ with the fuzzy relation $\tilde{R}$, which is known implication relation (IF-THEN rule) is used i.e.

$$\tilde{A}' = \tilde{R} \circ \tilde{B}'$$

In terms of membership function,

$$\mu_{\tilde{A}'}(y) = \max_{y \in Y}[\min[\mu_{\tilde{B}'}(y), \mu_{\tilde{R}}(x, y)]]$$

A typical problem in this fuzzy approximate reasoning is as follows:

Implication:	"If Rabi is much younger then he can work more"
Fact:	"Rabi cannot work much"
Conclusion:	"Rabi is not so young"

Fuzzy Inference

Generalized Hypothetical Syllogism: We now take up the extension of the classical Hypothetical Syllogism to generalized fuzzy hypothetical syllogism based on two conditional fuzzy production rules. This may be stated as follows:

Implication 1:	IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Implication 2:	y is $\tilde{B}'$ THEN z is $\tilde{C}$
Conclusion:	x is $\tilde{A}$ THEN z is $\tilde{C}'$

For the fuzzy sets $\tilde{A}$, $\tilde{B}$, $\tilde{B}'$, $\tilde{C}$ and $\tilde{C}'$ and fuzzy propositions (implications 1, 2), the new fuzzy proposition in the conclusion is inferred such that closer the set $\tilde{B}$ to $\tilde{B}'$, the closer the set $\tilde{C}'$ to $\tilde{C}$.

Fuzzy Inference

To compute the membership function of the fuzzy relation corresponding to the conclusion, $\tilde{R}_c$, the max-min composition of fuzzy relations corresponding to the implications 1 and 2, $\tilde{R}_1$ and $\tilde{R}_2$, is used i.e.

$$\tilde{R}_c = \tilde{R}_1 \circ \tilde{R}_2$$

In terms of membership function,

$$\mu_{\tilde{R}_c}(x, z) = \max_{y \in Y}[\min[\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z)]]$$

A typical problem in this fuzzy approximate reasoning is as follows:

Implication 1:	"If tomato is ripe then it taste sweet"
Implication 2:	"The sweet tomato is much red"
Conclusion:	"The ripe tomato is red"

Defuzzification methods

Defuzzification refers to the way of a crisp value expected from fuzzy sets as a representative. In general, there are seven methods used for defuzzifying the fuzzy output functions. They are:

(1) *Max-membership principle:* This method is given by the expression,

$$\mu_{\tilde{C}}(z^*) \geq \mu_{\tilde{C}}(z) \text{ for all } z \in Z.$$

This method is also referred as height method.

(2) *Centroid method:* This is the most widely used method. This can be called as center of gravity or center of area method. It can be defined by the algebraic expression

$$z^* = \frac{\int_Z \mu_{\tilde{C}}(z) z dz}{\int_Z \mu_{\tilde{C}}(z) dz}$$

$\int$ is used for algebraic integration.

Defuzzification methods

(3) *Weighted average method:* This method cannot be used for asymmetrical output membership functions, can be used only for symmetrical output membership functions. Weighting each membership function in the obtained output by its largest membership value forms this method. The evaluation expression for this method is

$$z^* = \frac{\sum \mu_{\tilde{C}}(\bar{z})\bar{z}}{\sum \mu_{\tilde{C}}(\bar{z})}$$

$\sum$ is used for algebraic sum.

(4) *Mean-max membership:* This method is related to max-membership principle, but the present of the maximum membership need not be unique, i.e., the maximum membership need not be a single point, it can be a range. This method is also called as middle of maxima method the expression is given as

$$z^* = \frac{a+b}{2}$$

where $a \times b$ are the end points of the maximum membership range.

(5) Centre of sums,

(6) Centre of largest area, and

(7) First of maxima or last of maxima

Fuzzy Inference System

The fuzzy inference system is a popular computing framework based on the concepts of fuzzy set theory, fuzzy IF-THEN rules and fuzzy reasoning. It has found successful applications in a wide variety of fields such as automatic control, data classifications, decision analysis, expert systems, robotics and patterns recognition. Because of its multidisciplinary nature, the fuzzy inference system (FIS) is known by numerous other names, such as fuzzy-rule base system, fuzzy expert system, fuzzy logic controller, and fuzzy associative memory. This is a major unit of a fuzzy logic system.

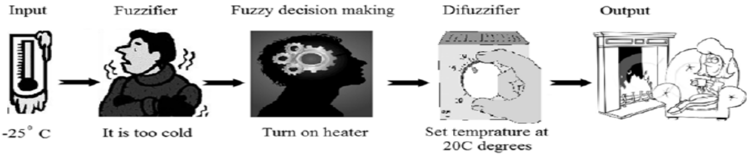

Fig. 1: Example of FIS.

Fuzzy Inference System

The basic FIS can take either fuzzy inputs or crisp inputs, but the outputs it produces are almost always fuzzy sets. When the FIS is used as a controller, it is necessary to have a crisp output. Therefore in this case defuzzification method is adopted to best extract a crisp value that best represents a fuzzy set. The Fig. 1 illustrates the overview of FIS.

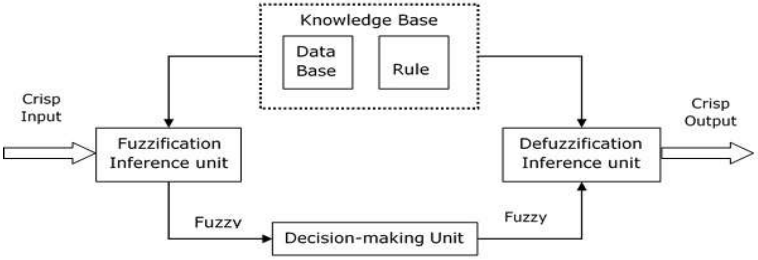

Fig. 2: Fuzzy Inference System.

Working of Inference System

Fuzzy inference system consists of a fuzzification interface, a rule base, a database, a decision-making unit, and finally a defuzzification interface. A FIS with five functional block described in Fig. 2. The function of each block is as follows:

- a fuzzification interface which transforms the crisp inputs into degrees of match with linguistic values;
- a rule base containing a number of fuzzy IF-THEN rules;
- a database which defines the membership functions of the fuzzy sets used in the fuzzy rules;
- a decision-making unit which performs the inference operations on the rules; and
- a defuzzification interface which transforms the fuzzy results of the inference into a crisp output.

The working of FIS is as follows. The crisp input is converted in to fuzzy by using fuzzification method. After fuzzification the rule base is formed. The rule base and the database are jointly referred to as the knowledge base. Defuzzification is used to convert fuzzy value to the real world value which is the output.

Fuzzy Inference Methods

The most important three types of fuzzy inference methods are (i) Mamdani's fuzzy inference method (ii) Takagi-Sugeno (TS) fuzzy inference method (iii) Tsukamoto fuzzy inference method, which have been widely employed in various applications. The differences between these three fuzzy inference methods lie in the consequent of the fuzzy rules and thus their aggregation and defuzzification procedures differ accordingly.

Mamdani FIS: Ebsahim Mamdani proposed this system in the year 1975 to control a steam engine and boiler combination by synthesizing a set of fuzzy rules obtained from peoples working on the system. In this case, the output membership functions are expected to be fuzzy sets. After aggregation process, each output variable contains a fuzzy set, hence defuzzification is important at the output stage. The following steps have been followed to compute the output from this inference method.

Mamdani FIS

1. Determining a set of fuzzy rules
2. Fuzzifying the inputs using the input membership functions
3. Combining the fuzzified inputs according to the fuzzy rules to establish a rule strength
4. Finding the consequence of the rule by combining the rule strength and the output membership function
5. Combining the consequences to get an output distribution
6. Defuzzifying the output distribution (this step is only if a crisp output (class) is needed).

The fuzzy rules are formed using "IF-THEN" statements and "AND/ OR" connectives. The consequence of the rule can be obtained in two steps:

(i). Computing the rule strength by combining the fuzzified inputs using the fuzzy combination process.

(ii). Clipping the output membership function at the rule strength

Mamdani FIS

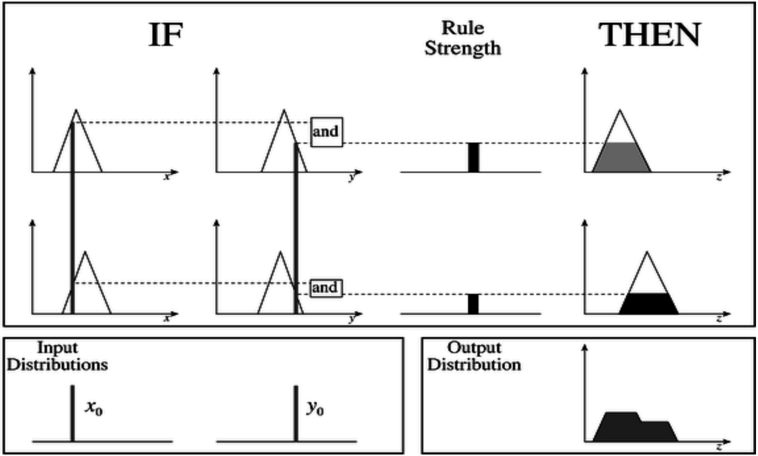

Fig. 3: Mamdani FIS.

Takagi-Sugeno FIS

The Takagi-Sugeno fuzzy model was proposed by Takagi, Sugeno, and Kang in the year 1985. This method was advice in an effort to develop a systematic approach to generating fuzzy rules from a given input-output data set. Sugeno-Takagi fuzzy model is also know as Sugeno model. A typical fuzzy rule in a Sugeno fuzzy model has the format

$$\text{IF x is } \tilde{A} \text{ AND y is } \tilde{B} \text{ THEN } z = f(x, y)$$

Where $\tilde{A}$, $\tilde{B}$ are fuzzy sets in antecedent and $z = f(x, y)$ is a crisp function in the consequent. Generally, $f(x, y)$ is a polynomial function for the inputs x and y, but it can be any general function as long as it describes the output of the system within the fuzzy region specified in the antecedent of the rule to which it is applied. When $f(x, y)$ is a constant the inference system is called a zero-order Sugeno model, which is a special case of the Mamdani system in which each rule's consequent is specified as a fuzzy singleton. When $f(x, y)$ is a linear function of x and y, the inference system is called a first-order Sugeno model.

Takagi-Sugeno FIS

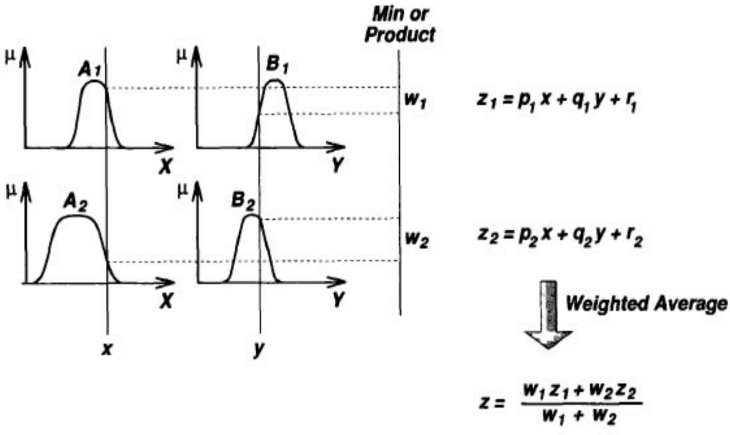

Fig. 4: Takagi-Sugeno FIS.

Takagi-Sugeno FIS

The main steps of the fuzzy inference process namely,

1. Fuzzifying the inputs-Here, the inputs of the system are made fuzzy.
2. applying the fuzzy operators-In this step, the fuzzy operators must be applied to get the output.

are exactly same.

Advantages of Sugeno and Mamdani Method:

- **Sugeno Method:**
  1. It is computationally efficient.
  2. It works well with linear techniques.
  3. It works well with optimization and adaptive techniques.
  4. It has guaranteed continuity of the output surface.
  5. It is well suited to mathematical analysis.
- **Mamdani Method:**
  1. It is intuitive.
  2. It has widespread acceptance.
  3. It is well suited to human input.

Tsukamoto FIS

In the Tsukamoto fuzzy inference model, the consequent of each fuzzy IF-THEN rule is represented by a fuzzy set with a monotonical membership function. As a result, inferred output of each rule defined as a crisp value induced by the rule of firing strength. The overall output is taken as the weighted average of each rule's output. Since each rule infers a crisp output, the Tsukamoto fuzzy model aggregates each rule's output by method of weighted average and thus avoids the time consuming process of defuzzification. However, the Tsukamoto fuzzy model is not used often. Since it is not transparent as either Mamdani or Sugeno fuzzy models.

Tsukamoto FIS

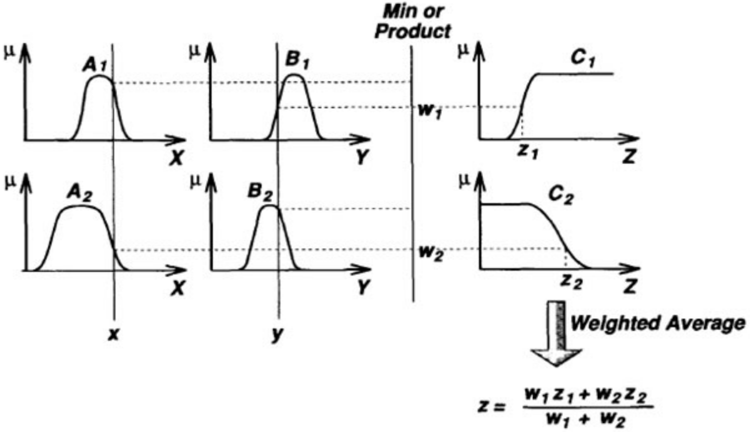

Fig. 5: Tsukamoto FIS.

Fuzzy Logic Control for Washing Machine

Consider washing time control of the washing machine, and design a fuzzy controller with two inputs and one output. Choose mud and axunge as the inputs and choose washing time as the output. We can define three fuzzy sets for mud and axunge, and define five fuzzy sets for washing time.
Consider MF for mud as SD (mud small), MD (mud middle), and LD (mud much) , the range of mud is in [0, 100], and the MF is designed as follows:

$$\mu_{mud} = \begin{cases} \mu_{SD}(x) = \frac{50-x}{50}, & 0 \le x \le 50 \\ \mu_{MD}(x) = \begin{cases} \frac{x}{50}, & 0 \le x \le 50 \\ \frac{100-x}{50}, & 50 \le x \le 100 \end{cases} \\ \mu_{LD}(x) = \frac{x-50}{50}, & 50 \le x \le 100 \end{cases} \tag{16}$$

Consider MF for axunge as NG (no axunge) , MG (middle axunge) , and LG (much axunge) , the range value of axunge is set as [0, 100], and the MF is designed as follows:

FIS for Washing Machine

$$\mu_{axunge} = \begin{cases} \mu_{NG}(y) = \frac{50-y}{50}, & 0 \leq y \leq 50 \\ \mu_{MG}(y) = \begin{cases} \frac{y}{50}, & 0 \leq y \leq 50 \\ \frac{100-y}{50}, & 50 \leq y \leq 100 \end{cases} \\ \mu_{LG}(y) = \frac{y-50}{50}, & 50 \leq y \leq 100 \end{cases} \tag{17}$$

and consider washing time as VS (very small), S (small), M (middle), L (long), and VL (very long), and the value is in the range of [0, 60]. MF of washing time is:

$$\mu_{time} = \begin{cases} \mu_{VS}(z) = \frac{10-z}{10}, & 0 \leq z \leq 10 \\ \mu_{S}(z) = \begin{cases} \frac{z}{10}, & 0 \leq z \leq 10 \\ \frac{25-z}{15}, & 10 \leq z \leq 25 \end{cases} \\ \mu_{M}(z) = \begin{cases} \frac{z-10}{15}, & 10 \leq z \leq 25 \\ \frac{40-z}{15}, & 25 \leq z \leq 40 \end{cases} \\ \mu_{L}(z) = \begin{cases} \frac{z-25}{15}, & 25 \leq z \leq 40 \\ \frac{60-z}{20}, & 40 \leq z \leq 60 \end{cases} \\ \mu_{VL}(z) = \frac{z-40}{20}, & 40 \leq z \leq 60 \end{cases} \tag{18}$$

FIS for Washing Machine

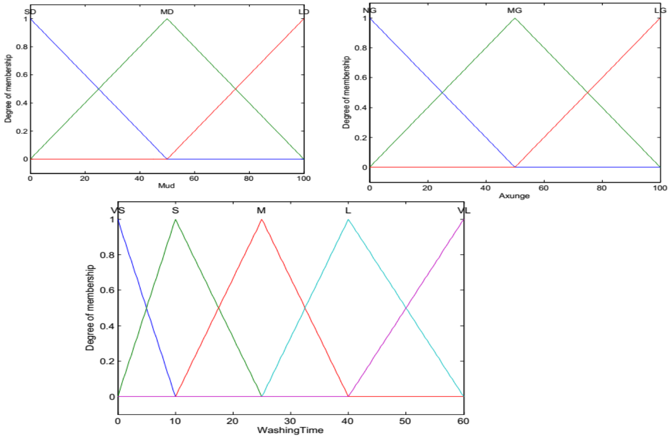

Fig. 6: Membership functions of input and output variables.

FIS for Washing Machine

Design fuzzy rules according to the input is mud and axunge, and the output is washing time. If design three membership functions for each input , then can design 9 rules. The format of the rule is

Table 2: Fuzzy rules of the washing machine

Washing time, z		Axunge, y		
		NG	**MG**	**LG**
Mud, x	**SD**	VS	S	M
	MD	M	M	L
	LD	L	L	VL

"IF Mud is $\tilde{A}$ AND Axunge is $\tilde{B}$ THEN Washing time is $\tilde{C}$"

which is shown in Table 2.

FIS for Washing Machine

If $x_0\,(mud) = 60$, $y_0, (axunge) = 70$, then

$$\mu_{SD}(60) = 0, \;\; \mu_{MD}(60) = \frac{4}{5}, \;\; \mu_{LD}(70) = \frac{1}{5}$$

$$\mu_{NG}(70) = 0, \;\; \mu_{MG}(70) = \frac{3}{5}, \;\; \mu_{LG}(70) = \frac{2}{5}$$

Then, four fuzzy rules are activated, and the results are shown in Table 3.

Table 3: Fuzzy rules of activation

Washing time, z		Axunge, y		
		$NG\,(0)$	$MG\,(\frac{3}{5})$	$LG\,(\frac{2}{5})$
Mud, x	$SD\,(0)$	0	0	0
	$MD\,(\frac{4}{5})$	0	$\mu_M(z)$	$\mu_L(z)$
	$LD\,(\frac{1}{5})$	0	$\mu_L(z)$	$\mu_{VL}(z)$

FIS for Washing Machine

From Table 3, four fuzzy rules are inspired as

Rule 1: IF x is MD and y is MG THEN z is M
Rule 2: IF x is MD and y is LG THEN z is L
Rule 3: IF x is LD and y is MG THEN z is L
Rule 4: IF x is LD and y is LG THEN z is VL

Since "AND" is used in the inference, then fuzzy intersection operator can be used and strength for premise of each fuzzy rule can be calculated as follows:

Strength of Rule 1 premise: $\min(4/5, 3/5) = 3/5$
Strength of Rule 2 premise: $\min(4/5, 2/5) = 2/5$
Strength of Rule 3 premise: $\min(1/5, 3/5) = 1/5$
Strength of Rule 4 premise: $\min(1/5, 2/5) = 1/5$

The inference of each fuzzy rule is shown in Table 4.

FIS for Washing Machine

Table 4: Inference of each fuzzy rule

Washing time, z		Axunge, y		
		$NG\,(0)$	$MG\,(\frac{3}{5})$	$LG\,(\frac{2}{5})$
Mud, x	$SD\,(0)$	0	0	0
	$MD\,(\frac{4}{5})$	0	$\min(3/5, \mu_M(z))$	$\min(2/5, \mu_L(z))$
	$LD\,(\frac{1}{5})$	0	$\min(1/5, \mu_L(z))$	$\min(1/5, \mu_{VL}(z))$

The inference of each fuzzy rule graphically shows in Fig. 7. Using centroid defuzzification method in output distribution, the washing time is 33.7 minute.

FIS for Washing Machine

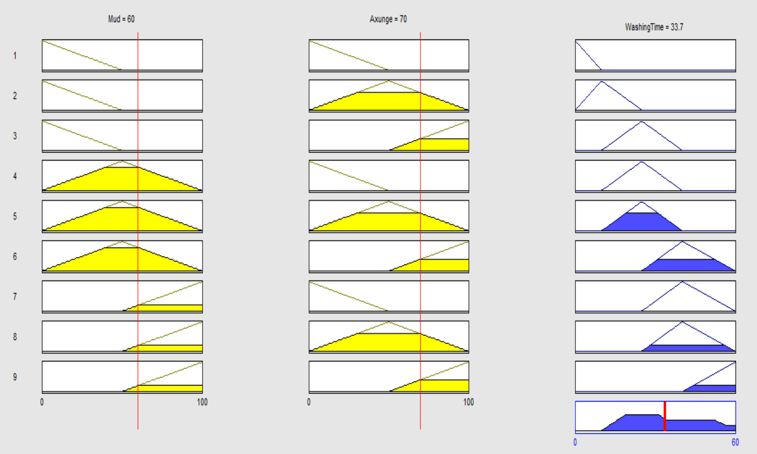

Fig. 7: Graphical representation of Rules.