Study Material

Dept. of Applied Mathematics
With Oceanology and Computer Eng.

Paper No. - MTM 205

Paper Name - Continuum Mechanics

Semester - 2

Topic of Lecturer : Motion in Two dimension, Concept of Stream function, and its different properties

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Lecturer No - 05

which $\Rightarrow \frac{\partial\varphi}{\partial x}=0, \frac{\partial\varphi}{\partial y}=0, \frac{\partial\varphi}{\partial z}=0$ everywhere

$\Rightarrow \varphi$ is independent of x, y, z

i.e., φ = constant everywhere.

$\therefore \varphi_1-\varphi_2$ = constant.

i.e., φ_1 and φ_2 can differ only by a constant. Therefore the velocity distribution given by φ_1 and φ_2 are identical and hence two motions are identical.

6.15 Motion in Two Dimensions :

Suppose a fluid moves in such a way that at any given instant the flow pattern in a certain plane is the same as that in all other parallel planes within the fluid. Then the flow is said to be two-dimensional.

If we take any one of the parallel planes to be the plane $z=0$, then at any point in the fluid having cartesian co-ordinates (x, y, z), all physical quantities (velocity, pressure, density, etc.) associated with the fluid are independent of z. Evidently in this case

$w=0$ and $u=u(x,y,t)$, $v=v(x,y,t)$ where $\vec{V}=(u,v,w)=(u,v,0)$.

6.16 Lagrange's Stream Function (Current function):

In case of two-dimensional motion, the differential equation of the stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}$$

i.e. $vdx-udy=0$ .... (i)

The equation of continuity for incompressible fluid, in two dimensions is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{.................. (ii)}$$

i.e. $$\frac{\partial v}{\partial y}=\frac{\partial(-u)}{\partial x}$$

Above result shows that (i) is an exact differential and let it will be $d\psi$, i.e.,

$$vdx-udy=d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy$$

which $\Rightarrow v=\frac{\partial\psi}{\partial x}, -u=\frac{\partial\psi}{\partial y}$ .... (iii)

Now, (i) takes the form as

$$d\psi = 0$$

Int., $\psi =$ constant .... (iv)

This function $\psi = \psi(x, y)$ is called the *stream function or current function.*

Since the stream lines are given by (i), so it follows that stream function is constant along the stream line.

Note-1. Stream function exists for all types of two dimensionsl motion-rotational or irrotational.

Note-2. The necessary conditions for the existence of ψ are:

i) the flow must be continuous,

ii) the flow must be incompressible.

Note-3. The existence of a stream function is a consequence of stream lines and equation of continuity for incompressible fluid.

Note-4. φ and ψ are conjugate functions.

Proof. For irrotational fluid motion we have

$$\vec{V} = -\vec{\nabla}\varphi \text{ where } \varphi \text{ is velocity potential}$$

$$\Rightarrow u = -\frac{\partial \varphi}{\partial x}, v = -\frac{\partial \varphi}{\partial y}$$

If ψ is a stream function, then

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

$$\therefore \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

So, $$\nabla^2\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) = 0$$

$$\nabla^2\psi = 0 \text{ .... (i)}$$

Again, $u = -\frac{\partial \varphi}{\partial x}, v = -\frac{\partial \varphi}{\partial y}$ and equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

$$\therefore \frac{\partial}{\partial x}\left(-\frac{\partial \varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \varphi}{\partial y}\right)=0$$

$$-\left[\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}\right]=0$$

i.e. $\nabla^2\varphi = 0$ .... (ii)

(i) and (ii) implies that φ and ψ both satisfies Laplace's equation i.e., φ and ψ are conjugate functions.

Note-5 Existence of φ and ψ:

i) The stream function ψ exists whether the motion is irrotational or not.

ii) The velocity potential φ exists only when the motion is irrotational.

iii) When motion is irrotational, φ exists.

iv) φ and ψ both satisfy Laplace's equation and

$$\frac{\partial \varphi}{\partial x}=\frac{\partial \psi}{\partial y},\ \frac{\partial \varphi}{\partial y}=-\frac{\partial \psi}{\partial x}.$$

Note-6 The family of curves $\varphi(x,y)=$ constant and $\psi(x,y)=$ constant, cut orthogonally at their points of intersection.

Proof. $\varphi(x,y)=$ constant $d\varphi=0$

i.e., $$\frac{\partial \varphi}{\partial x}dx+\frac{\partial \varphi}{\partial y}dy=0$$

$$\frac{dy}{dx}=-\frac{\varphi_x}{\varphi_y}=m_1\text{, say}$$

which is the gradient of tangent to the curve $\varphi=$ constant.

Again, $\psi(x,y)=$ constant $\Rightarrow d\psi=0$

i.e. $$\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=0$$

$$\therefore \frac{dy}{dx}=-\frac{\psi_x}{\psi_y}=m_2\text{, say}$$

which is the gradient of tangent to the curve $\psi=$ constant.

Now, $m_1 \times m_2 = \left(-\frac{\varphi_x}{\varphi_y}\right) \times \left(-\frac{\psi_x}{\psi_y}\right) = \frac{u}{v} \times \frac{v}{(-u)} = -1$

Hence the curves of constant potential and constant stream functions cut orthogonally at their points of intersection.

(ii) the flow must be incompressible.

3·3. *The difference of the values of ψ at the two points represents the flux of a fluid across any curve joining the two points.*

Proof. Suppose ds is a line element at a point $P(x, y)$ of a curve AB. Let the tangent PT make an angle θ with x-axis. Let PN be normal at P and (u, v) the velocity components of the fluid at P. Direction cosines of the normal PN are

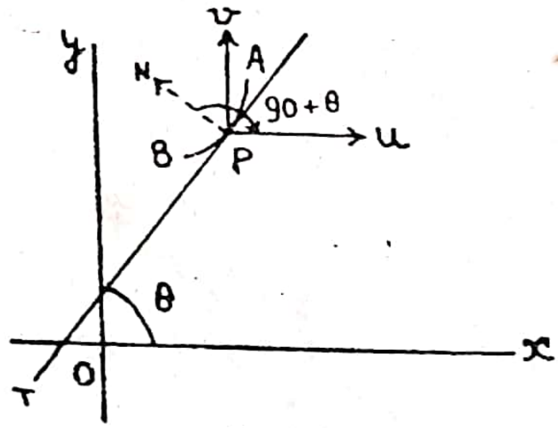

Fig. 20

$\cos(90+\theta)$, $\cos\theta$, $\cos 90$,

i.e., $-\sin\theta$, $\cos\theta$, 0.

(For PN makes angles $90+\theta$, θ, 90 with x, y, z axes respectively

Inward normal velocity$=\hat{n}.q$, in usual notation

$$=u(-\sin\theta)+v(\cos\theta)+(0).0$$

$$=-u\sin\theta+v\cos\theta.$$

Flux across the curve AB from right to left
=density.normal velo.area of the cross section

$$=\int_{AB}\rho(\hat{n}.q)\,ds=\int_{AB}\rho(-u\sin\theta+v\cos\theta)\,ds$$

$$=\rho\int_{AB}\left[-u\frac{dy}{ds}+v\frac{dx}{ds}\right]ds \text{ as } \tan\theta=\frac{dy}{dx}$$

$$=\rho\int_{AB}\left[\left(\frac{\partial\psi}{\partial y}\right)dy+\left(\frac{\partial\psi}{\partial x}\right)dx\right]=\rho\int_{AB}d\psi=\rho\,[\psi_B-\psi_A]$$

where ψ_A and ψ_B are the values of ψ at A and B respectively.

Flux across AB is $\rho\,[\psi_B-\psi_A]$.

This proves the required result.