M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Semester-IV

Paper-MTM402 Unit-1

Fuzzy Mathematics with Applications

(Application Fuzzy Sets)

Unit Structure:

3.1 Introduction

In the crisp linear programming problem, the aim is to maximize or minimize a linear objective function subject to some linear constraints. But in many real life practical situations the LPP can not be specified precisely. The objective function and/or the constraint functions appears in the problem in the fuzzy sense having a vague meaning. To handle such problems fuzzy linear programming problem is introduced.

In such problems the decision maker has more flexibility. Fuzziness may occur in a linear programming problem in many ways. The objective function may be fuzzy, the inequalities may he fuzzy or the problem parameters e, A, b may be in terms of fuzzy numbers. Different methods are there to solve fuzzy LPP depending on the character of fuzziness. Some of them will be discussed in detail in this module.

3.2 Classification of fuzzy LPP

The crisp linear programming problem may be stated as Optimize $Z = CX$
subject to the constraints $Ax \leq=\geq b$

and $x \geq 0$

where $C \in R^n$, $b^T \in R^m$, $x^T \in R^n$ and A is $m \times n$ real matrix. We shall use the following notations to represents fuzzy quantities.
$\tilde{z}$ for fuzzy objective
$\tilde{b}$ for fuzzy resource
$\tilde{c}$ for fuzzy costs
$\tilde{A}$ for fuzzy coefficients matrix
$\lessapprox$ for fuzzy inequality.
In a fuzzy LPP, the fuzzy environment may occur in the following possible ways

(i) Instead of maximizing or minimizing the objective function the decision maker needs to achieve some aspiration level which itself may not even be definable crisply. As for example the decision maker may have a target to "improve the present sales situation considerably".

(ii) The constraints appeared in the LPP might be vague. The inequalities " $\leq$ *or* $=$ *or* $\geq$ "may not mean in the strict mathematical sense. Some violations 'may be acceptable within some tolerance limit. As for example the decision maker might say "try to contact about 1800 customers per week and it must not be less than 1600 customers per week in any situation".

(iii) The components of the cost vector c, the requirement vector b and the coefficient matrix A may not be crisp numbers instead some or all of them may be fuzzy numbers. The inequalities in such situation may be interpreted in terms of ranking of fuzzy numbers.

The class of fuzzy LPP can be broadly classified as follows .

(i) LPP with fuzzy inequalities and crisp objective function.

(ii) LPP with crisp inequalities and fuzzy objective function.

(iii) LPP with fuzzy inequalities and fuzzy objective function.

(iv) LPP with fuzzy resources and fuzzy coefficient i.e. LPP with fuzzy parameters i.e, elements of c, b and A are fuzzy numbers.

We have noted that there are different types of fuzzy LPP. Depending on the types of the fuzzy LPP the methods of solving them are also different. The following table shows the types of the fuzzy LPP and the standard available method for solving them.

Types	**Methods**
1) Crisp LPP	Simplex method
2)$\tilde{b}$	i)Parametric Programming ii) Verdegay's method iii) Chana's method .
3)$\tilde{z}$ and $\tilde{b}$	i)Wemer's method ii)Zimmermann's method iii)Lai and Hwang's method
4)$\tilde{c}$	Parametric Programming
5)$\tilde{A}$ 6)$\tilde{b}$ and $\tilde{c}$ 7)$\tilde{A}$ and $\tilde{b}$ 8)$\tilde{A}$ and $\tilde{c}$ 9)$\tilde{A}$ and $\tilde{b}$ and $\tilde{c}$	Carlsson and Korhonen's method
10)$\tilde{Z}$ and $\tilde{A}$ 11)$\tilde{Z}$ and $\tilde{A}$ and $\tilde{b}$	Lai and Hwang's method

3.3 Bellman and Zadeh's Principle

Let the fuzzy environment has a set of p goals G_1, G_2,..... G_p along with a set of n constrain C_l, C_2,..... C_n and each of them is expressed by fuzzy sets on the universal set X. For such a model decision making, Bellman and Zadeh proposed that a fuzzy decision is determined by an appropriate, aggregation of the fuzzy sets $\tilde{G}_1, \tilde{G}_2 \ldots \tilde{G}_p$ and$\tilde{C}_1, \tilde{C}_2, \ldots \tilde{C}_n$. In this approach the symmetry between goals and constraints is the main feature. Bcllrnann and Zadeh suggested the aggregation operator be the fuzzy intersection. The fuzzy decision $\tilde{D}$ is defined as the intersection of all $\tilde{G}_i$ and $\tilde{C}_j$ i.e$\tilde{D} = (\tilde{G}_1 \bigcap \tilde{G}_2 \bigcap \ldots \bigcap \tilde{G}_p) \bigcap (\tilde{C}_1 \bigcap \tilde{C}_2 \bigcap .... \bigcap \tilde{C}_p)$. The membership function of $\tilde{D}$ is given by

$\mu_{\tilde{D}}(x) = min\{\mu_{\tilde{G}_1}(x), \mu_{\tilde{G}_2}(x), \ldots \mu_{\tilde{G}_p}(x), \mu_{\tilde{C}_1}(x, \mu_{\tilde{C}_2}(x)), \mu_{\tilde{C}_n}(x)\}$ Once the fuzzy decision $\tilde{D}$ is found. the optimal decision $\dot{x}$ is determined as $\dot{x} \in X$ satisfying $\mu_{\tilde{D}}(\dot{x}) = \max_x \mu_{\tilde{D}}(x)$

3.3.1 Illustration of Bellman and Zadebts Principle

Zimmennann considered the fuzzy decision problem in which we are to find a real number which is in the vicinity of 15 and is substantially larger than 10. The constaint of the point lying in the vicinity of 15 may be regarded as a fuzzy constraint $\tilde{C}$ and the goal of having its value larger than 10 is regarded as a fuzzy goal $\tilde{G}$. Let us e the membership function of $\tilde{C}$ and $\tilde{G}$ as follows.

$$\mu_{\tilde{C}}(x) = \{1 + (x - 15)^4\}^{-1}$$

$$\mu_{\tilde{G}}(x) = \begin{cases} 0 & \text{for} x \leq 10 \\ 1 + (x - 10)^{-2}\}^{-1} & \text{for} x > 10 \end{cases}$$

By the principle of Bellman and Zadeh, the fuzzy decision $\tilde{D}$ is given by $\tilde{C} \bigcap \tilde{G}$. The membership function

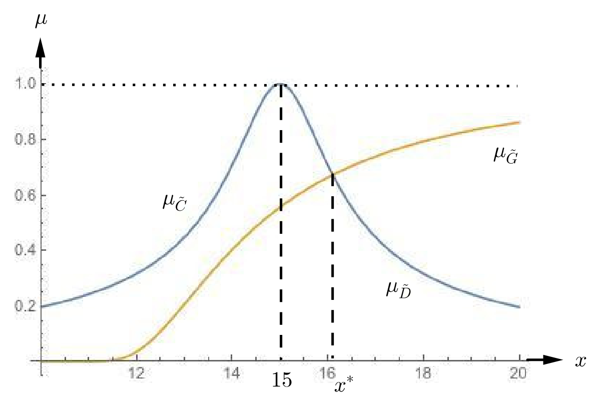

Figure 3.1: Fuzzy optimal decision.

of $\tilde{D}$ is given by
$\mu_{\tilde{D}}(x) = min\{\mu_{\tilde{C}}(x), \mu_{\tilde{G}}(x)\}$ To get the optimal decision $\dot{x}$ we proceed as follows . $\mu_{\tilde{D}}(\dot{x}) = \max_x \mu_{\tilde{D}}(x)$
For $\alpha \in (0, 1)$ first determine all points for which $\mu_{\tilde{D}}(x) \geq \alpha$. These decisions x satisfying $\mu_{\tilde{D}}(x) \geq \alpha$ will have at least α degree of membership value. So particular $\dot{x}$ for which α becomes maximum will be the required optimal decision (as for α maximum $\mu_{\tilde{D}}(x)$ will also become maximum).
Hence the optimal decision $\dot{x}$ is the solution of the problem:

$$\begin{aligned} \text{Maximize} \quad & \alpha \\ \text{subject to} \quad & \mu_{\tilde{G}_i}(x) \geq \alpha \\ & \mu_{\tilde{C}_i}(x) \geq \alpha \\ & 0 \leq \alpha \leq 1, \\ & x \geq 0, \end{aligned}$$

3.3.2 Another classification of fuzzy LPP

The class of fuzzy LP can be classified also as

(i) Symmetric fuzzy LPP and

(ii) Non symmetric fuzzy LPP.

Symmeric fuzzy LPP : The symmetric models are based on the definition of fuzzy decision as proposed by Bellman and Zadeh. The basic feature here is the symmetry of objectives and constraints. The decision set here is obtained as the intersection of the fuzzy sets corresponding to the objectives and constraints.
Non Symmetric fuzzy LPP : In the non-symmetric fuzzy LPP the constraints and the objectives are regarded as distinct entity. There are two approaches for non-symmetric model. In the first approach a fuzzy set of decisions is determined first and then the crisp objective function is optimized over this fuzzy set of decisions, This approach leads to a parametric LPP. In the second approach also a fuzzy set or decisions is determined first and then a suitable membership function is determined for the objective function. The problem is then solved as the symmetric case.

3.4 Verdegay's approach to solve fuzzy LPP

Verdegay considered the fuzzy LPP where the inequality is fuzzy or the resource is fuzzy. The general model of fuzzy LPP with fuzzy inequality is

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & (Ax)_i \leq b_i, i = 1, 2, \dots m \\ & x \geqslant 0, \end{aligned}$$

Here the fuzzy constraint $(Ax)_i \leq b_i$ has the meaning that the constraint $(Ax)_i \leq b_i$, is absolutely satisfied, whereas the constraint $(AX)_i > b_i + P_i$ is absolutely violated, Here P_i is the maximum tolerance from P_i as determined by the decision maker.

The general model of fuzzy LPP with fuzzy resources is

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & (Ax)_i \leq \tilde{b_i}, i = 1, 2, \dots m \\ & x \geqslant 0, \end{aligned}$$

where b_i for all i are in $[b_i, b_i + p_i]$ with given P_i.
If in both the LPP with fuzzy constraints and in fuzzy resources the tolerance limit Pi is same and both the LPP has same membership function then Vardegay proved that both the problems are equivalent.
Verdagay showed that this fuzzy LPP is equivalent 10 a crisp parametric LPP. The fuzzy constraint. or the fuzzy resources arc transformed into crisp constraint by choosing appropriate membership function for each

constraint. Here $(Ax)_i \in [b_i, b_i + p_i]$ and the membership function is taken as a monotonically decreasing function and the decrease is taken along a linear function. Thus the membership function corresponding to the ith constraint is taken as .

$$\mu_i(x) = \begin{cases} 1 & \text{for } (Ax)_i \leq b_i \\ \{b_i + p_i - (Ax)_i\}/p_i & \text{for } \; b_i \leq (Ax)_i \leq b_i + p_i \\ 0 & \text{for } (Ax)_i > b_i + p_i \end{cases}$$

The crisp LPP equivalent to this fuzzy LPP is taken as

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & \mu_i(x) \geq \quad \alpha \\ & 0 \leq \alpha \leq 1, \\ & x \geq 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & (Ax)_i \leq \; b_i + (1-\alpha)p_i \\ & x \geqslant 0, \\ & 0 \leq \alpha \leq 1, \end{aligned}$$

This LPP is a standard parametric LPP with $\theta = 1$- α as parameter. So the solution of the given fuzzy LPP is obtained by solving this equivalent crisp parametric LPP. Here, we note that we have an optimal solution for each $\alpha \in [0, 1]$. So the solution with α grade of membership is actually fuzzy. Also we note that this is a non-symmetric model. To develop the idea of fuzzy LPP we consider the following problem. Also the notion of determining the membership function of fuzzy constraint will be clear from this example.

3.4.1 **Example.**

Three metals namely iron, copper and zinc are required to produce two alloys A and
B. To produce 1 metre rod of A, 1 kg iron, 1 kg copper and 0.5 kg zinc and to produce 1 metre rod of B. 1 kg copper and 1 kg zinc are needed. Total available quantities of'metals ranges as follows
iron: 3 kg to 9 kg, copper: 4 kg to 8 kg and
zinc: 3 kg to 5 kg. The profits of selling one unit of A and B are respectively Rs. 2 and Re 1.
Find the maximum profit.

Solution. All informations of the problem can be put in the following table.

Alloy	A	B	Availability quantity
Iron	1 kg	0 kg	3 kg to 4kg
copper	1 kg	1 kg	4 kg to 6kg
Zinc	0.5 kg	1 kg	3 kg to 5 kg
Profit	Rs. 2	Re 1	

Here the available quantities of the metals are not a fixed amount, they are given in a range. So the problem is not a crisp problem, it becomes a fuzzy problem. To formulate this problem as a LPP, let x_i metre of alloy A and x_2 metre of alloy B be produced. Then the fuzzy LPP becomes

$$\begin{aligned} &\text{Maximum } z = 2x_1 + x_2 \\ &\text{subject to } \ \mathrm{x}_1 + \ 0x_2 \leq \ 3 \text{ or } 4 \\ &\qquad\qquad x_1 + \ x_2 \leq \ 4 \text{ or } 6 \\ &\qquad 0.5x_1 + \ x_2 \leq \ 3 \text{ or } 5 \\ &\qquad x_j \geq 0, \quad j = 1, 2. \end{aligned}$$

3.4.2 **Membership function of the ith constraint**

The graph of the LPP with lower limits of the available quantities of iron, copper and zinc i.e. 3 kg iron, 4 kg copper and 3 kg zinc. Also lines are drawn with quantities as upper limits i.e. 4 kg iron, 6 kg copper and 5 kg zinc. The thick lines AB, BC, CD represents respectively the lower limits i.e. 3 kg iron, 4 kg copper and 3 kg zinc whereas the dotted lines $\acute{A}\acute{B}$, $\acute{B}\acute{C}$ and$\acute{C}\acute{D}$ represents respectively the upper limits i.e. 4 kg iron, 6 kg copper and 5 kg zinc.

the line DM represents zinc = 3 kg and the line $\acute{D}\acute{M}$ represents zinc = 5 kg. So in the region ODM zinc $\leq$ 3 kg which is always available and hence in this region the membership function $\mu_3(x)$ should have a value 1. Again in the region beyond the line $\acute{D}\acute{M}$,amount of zinc is more than 5 kg which is not available, hence in this region the membership function should have a value zero. In the region between the lines DM and $\acute{D}\acute{M}$, the value of the membership function should lie in the interval $(0, 1)$, as the availability of zinc there is in between 3 kg to 5 kg which is a doubtful situation. The membership function $\mu_3(x)$ should change its value there linearly from 1 on DM to on $\acute{D}\acute{M}$. Hence the membership function $\mu_3(x)$ is defined as

$$\mu_3(x) = \begin{cases} 1 & for\ x \in region\ ODM \\ (5 - x)/2 & for\ x \in region\ DM\acute{D}\acute{M} \\ 0 & for\ x \in beyond\ \acute{D}\acute{M} \end{cases}$$

i.e.

$$\mu_3(x_1, x_2) = \begin{cases} 1 & for\ 0.5x_1 + x_2 \leq 3 \\ (5 - 0.5x_1 - x_2)/2 & for\ 3 < 0.5x_1 + x_2 < 5 \\ 0 & for\ 0.5x_1 + x_2 \geq 5 \end{cases}$$

Similarly, the membership function J.ll (x) corresponding to the metal iron and f.L2 (x) corresponding to the metal copper are defined as follows

$$\mu_1(x_1, x_2) = \begin{cases} 1 & for\ 0.5x_1 \leq 3 \\ (4 - x_1) & for\ 3 < x_1 < 4 \\ 0 & for\ x_1 \geq 4 \end{cases}$$

$$\mu_2(x_1, x_2) = \begin{cases} 1 & for\ x_1 + x_2 \leq 4 \\ (6 - x_1 - x_2)/2 & for\ 4 < x_1 + x_2 < 6 \\ 0 & for\ x_1 + x_2 \geq 6 \end{cases}$$

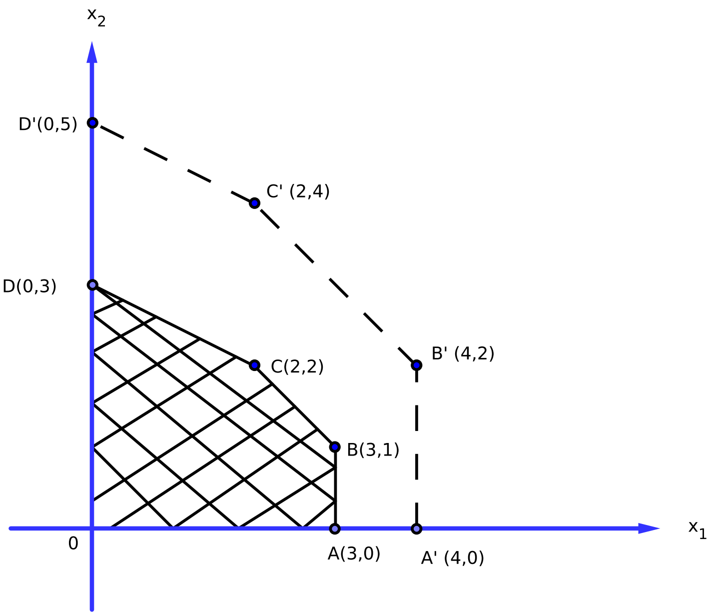

Figure 3.2: Fuzzy region of fuzzy LPP.

To discuss Verdegay's approach we.considerthe following example.

3.4.3 Example.

A company produces four items A, B, C and D. The inputs for the production are man-weeks, material X and material Y. The availability of the resources and profits corresponding to the items A, B and$C$ are shown in the table below. Using Verdegay's method find its solution.

Item	Man Weeks	Material X	Material Y	Unit Profit
A	1	7	3	4
B	1	5	5	5
C	1	3	10	9
D	1	2	15	11
Availability	15 to 18	120	100 to 120	Maximize

Solution. Here the availability of the material X is 120 unit which is a precise quantity. But the available total man-weeks and material Y are imprecise and their maximum tolerances are respectively 3 and 20 units respectively as 18-15=3 and 120-100=20. Let x_1,x_2,x_3 and x_4, be the amount produced for the items A, B, C and D respectively. Then the problem can be formulated as the following fuzzy LPP.

$$
\begin{aligned}
&\text{Maximum } z = 4x_1 + 5x_2 + 9x_3 + 11x_4 \\
&\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 \leq 15 \text{ to } 18 \\
&\quad 7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120 \\
&\quad 3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100 \text{ to } 120 \\
&\quad x_j \geq 0, \quad j = 1, 2, 3, 4.
\end{aligned}
$$

$$\begin{aligned}
g_1(x) &= x_1 + x_2 + x_3 + x_4 \\
g_2(x) &= 7x_1 + 5x_2 + 3x_3 + 2x_4 \\
g_3(x) &= 3x_1 + 5x_2 + 10x_3 + 15x_4 \\
cx &= 4x_1 + 5x_2 + 9x_3 + 11x_4
\end{aligned}$$

Hence the problem becomes

$$\begin{aligned}
\text{Maximize} \quad & z = cx \\
\text{subject to} \quad & g_1(x) \leq \ 15 \ \ \textit{to} \ \ 18 \\
& g_2(x) \leq \ 120 \\
& g_3(x) \leq \ 100 \ \ \textit{to} \ \ 120 \\
& x \geqslant 0,
\end{aligned}$$

The membership functions of the first and third constraints are given by

$$\mu_1(x) = \begin{cases} 1 & \text{for} \ \ g_1(x) \leq 15 \\ (18 - g_1(x))/3 & \text{for } 15 < g_1(x) < 18 \\ 0 & \text{for} \ \ g_1(x) \geq 18 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{for} \ \ g_1(x) \leq 100 \\ (120 - g_3(x))/3 & \text{for } 100 < g_3(x) < 120 \\ 0 & \text{for } g_3(x) \geq 120 \end{cases}$$

Using Verdegay's method the crisp parametric programming problem equivalent to the given fuzzy LPP is given by

$$\begin{aligned}
\text{Maximize} \quad & z = cx \\
\text{subject to} \quad & \mu_1(x) \leq \ \alpha \\
& g_2(x) \leq \ 120 \\
& \mu_3(x) \leq \ \alpha \\
& 0 \leq \alpha \leq 1 \\
& x \geqslant 0,
\end{aligned}$$

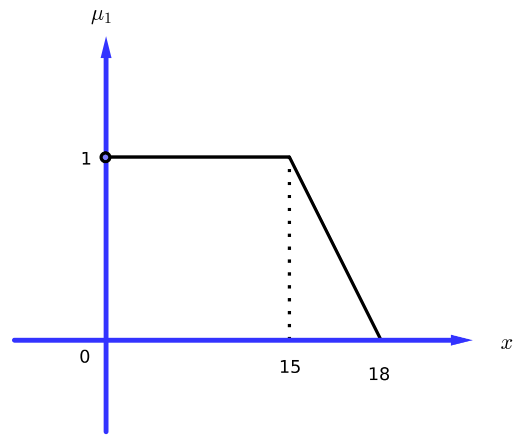

Figure 3.3: Non convex and normal fuzzy set.

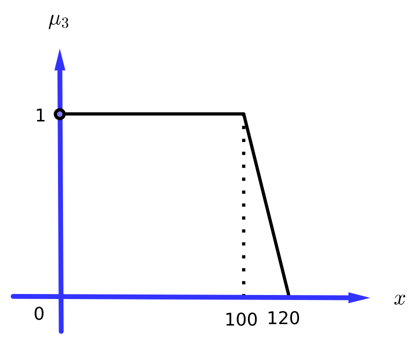

Figure 3.4: Non convex and normal fuzzy set.

i.e.

$$\begin{aligned}
\text{Maximize} \quad & z = cx \\
\text{subject to} \quad & g_1(x) \leq \ 15 + (1-\alpha)3 \\
& g_2(x) \leq \ 120 \\
& g_3(x) \leq \ 100 + (1-\alpha)20 \\
& 0 \leq \alpha \leq 1 \\
& x \geqslant 0,
\end{aligned}$$

$$\begin{aligned}
\text{Max} \quad & z = 4x_1 + 5x_2 + 9x_3 + 11x_4 \\
\text{s.t.} \quad & x_1 \ + x_2 + x_3 + x_4 \ \leqslant \ 15 + 3\theta \\
& 7x_1 + 5x_2 + 3x_3 + 2x_4 \ \leqslant \ 120 + 0\theta \\
& 3x_1 + 5x_2 + 10x_3 + 15x_4 \ \leqslant \ 100 + 20\theta \\
& x_j \geqslant 0, \ j = 1, 2, 3, 4.
\end{aligned}$$

where $\theta = 1 - \alpha$ is the parameter and $0 \leq \theta \leq 1$. To solve this parametric programming problem we first solve the corresponding LPP obtained by taking $\theta = 0$ using simplex method. The tables are shown below where the variables x_5,x_6 and x_7 are slack variables.

		c	4	5	9	11	0	0	0	
c_B	x_B	b	y_1	y_2	y_3	y_4	y_5	y_6	y_7	min ratio
0	x_5	15	1	1	1	1	1	0	0	15
0	x_6	120	7	5	3	2	0	1	0	60
0	x_7	100	3	5	10	$\boxed{15}$	0	0	1	100/15
	$z=0$	z_j-c_j	-4	-5	-9	-11	0	0	0	
0	x_5	25/3	$\boxed{4/5}$	2/3	1/3	0	1	0	$-1/15$	125/12
0	x_6	320/3	33/5	13/5	5/3	0	0	1	$-/15$	1600/99
11	x_4	20/3	1/5	1/3	2/3	1	0	0	1/15	100/3
	$z=220/3$	z_j-c_j	$-9/5$	$-4/3$	$-5/3$	0	0	0	11/15	
4	x_1	125/12	1	5/6	5/12	0	5/4	0	$-1/12$	25
0	x_6	455/12	0	$-7/6$	$-13/12$	0	$-33/4$	1	5/12	
11	x_4	55/12	0	1/6	$\boxed{7/12}$	1	$-1/4$	0	1/12	55/7
	$z=1105/12$	z_j-c_j	0	1/6	$-11/12$	0	9/4	0	7/12	
4	x_1	50/7	1	5/7	0	0	$-5/7$	10/7	$0-1/7$	
0	x_6	325/7	0	$-6/7$	0	13/7	$-61/7$	1	4/7	
9	x_3	55/7	0	2/7	1	12/7	$-3/7$	0	1/7	
	$z=695/7$	z_j-c_j	0	3/7	0	11/7	13/7	0	5/7	

Using parametric programming technique the final table of this simplex method can be used to get the optimal values of the basic variables and the corresponding value of the objective function for the parametric LPP as follows.

The optimal values of the basic variables for the parametric LPP are given by

$x_B = b + 3\theta y_5 + 0\theta y_6 + 20\theta y_7$

$$\begin{bmatrix} x_1 \\ x_6 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{50}{7} \\ \\ \frac{325}{7} \\ \\ \frac{55}{7} \end{bmatrix} + 3\theta \begin{bmatrix} \frac{10}{7} \\ \\ \frac{-61}{7} \\ \\ \frac{-3}{7} \end{bmatrix} + 0\theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 20\theta \begin{bmatrix} \frac{-1}{7} \\ \\ \frac{4}{7} \\ \\ \frac{1}{7} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_6 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{50}{7} + \frac{10\theta}{7} \\ \\ \frac{325}{7} - \frac{103\theta}{7} \\ \\ \frac{55}{7} + \frac{11\theta}{7} \end{bmatrix}$$

The optimal value of the objective function is given by

$$\begin{aligned} \dot{Z} &= Z + 3\theta(Z_5 - C_5) + 0\theta(Z_6 - C_6) + 20\theta(Z_7 - C_7) \\ &= \frac{695}{7} + 3\theta(\frac{13}{7}) + 0 + 20\theta\frac{5}{7} \\ &= (695 + 139\theta)/7 \end{aligned} \tag{3.1}$$

Hence the optimal solution of the parametric LPP i.e. of the given fuzzy LPP is

$$\begin{aligned} x_1 &= (50+10\theta)/7 \\ x_2 &= 0 \\ x_3 &= (55+11\theta)/7 \\ x_4 &= 0 \\ and\ \ Z_{max} &= (695+139\theta)/7 \ \ where \ \ 0 \leq \theta \leq 1 \end{aligned} \tag{3.2}$$

We note that the answer depends on the choice of the value ofθ by the decision maker.

3.5 Werners' method for solving fuzzy LPP

The general fuzzy LPP with fuzzy inequality is

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & (Ax)_i \leq b_i, i = 1, 2, \ldots m \\ & x \geqslant 0, \end{aligned}$$

Wemers proposed that because of fuzzy inequality constraint its effect will fall on the objective unction and as a result the objective function should also be fuzzy.
Let the tolerances for the m constraints because of fuzzy inequalities be $p_1, p_2, \ldots, p_m$ So the lower and upper limits of the resources will be b_i, and $b_i + p_i$ for each $i = 1, 2, \ldots, m$. Here we note that the given fuzzy LPP may be given equivalently also as fuzzy resource lying in $(b_i, l_i + p_i)$.
:. The constraints $(Ax)_i \leq b_i, i = 1, 2, \ldots, m$ are satisfied completely and the constraints $(Ax)_i > b_i + p_i$, $i = 1, 2, \ldots, m$ are never satisfied. The constraints $(Ax)_i \leq b_i'$ where $b_i' \in (b_i, l_i + p_i)$ are satisfied partly. Thus the value of the membership function for should be 1 for $(Ax)_i \leq b_i'$,$b_i < b_i' < b_i + p_i$ should lie in (0, 1) and for $(Ax)_i > b_i + p_i$ it should be O. Hence the membership function for i^{th} constraint (i = 1,2,... m) is given by

$$\mu_i(x) = \begin{cases} 1 & \text{for } (Ax)_i \leq b_i \\ \{b_i + p_i - (Ax)_i\}/p_i & \text{for } \ b_i < (Ax)_i < b_i + p_i \\ 0 & \text{for } (Ax)_i > b_i + p_i \end{cases}$$

To construct the membership function for the objective function Wemers suggested to solve two LPP one with lower limit of resources and other with upper limit of resources. These two LPP s are thus

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & (Ax)_i \leq b_i, i = 1, 2, \ldots m \qquad ....(1) \\ & x \geqslant 0, \end{aligned}$$

and

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & (Ax)_i \leq b_i + p_i, i = 1, 2, \ldots m \qquad ....(2) \\ & x \geqslant 0, \end{aligned}$$

3.5.1 Example to explain Werners' method

Let the LPP with fuzzy resources be

$$\begin{aligned} \text{Max} \quad & Z = 4x_1 + 5x_2 + 9x_3 + 11x_4 \\ \text{s.t.} \quad & g_1(x) = x_1 + x_2 + x_3 + x_4 \leqslant \widetilde{15} \\ & g_2(x) = 7x_1 + 5x_2 + 3x_3 + 2x_4 \leqslant \widetilde{80} \\ & g_3(x) = 3x_1 + 5x_2 + 10x_3 + 15x_4 \leqslant \widetilde{100} \\ & x_j \geqslant 0, \; j = 1, 2, 3, 4. \end{aligned}$$

and the tolerances as $p_1 = 5$,4$p_2 = 40$,$p_3 = 30$. To get membership function for the objective function we have to solve two LPPs one with the. lower limits of fuzzy resources and other with the upper limits of fuzzy resources. These two LPP are follows.

$$\begin{aligned} \text{Max} \quad & Z = 4x_1 + 5x_2 + 9x_3 + 11x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \leqslant 15 \\ & 7x_1 + 5x_2 + 3x_3 + 2x_4 \leqslant 80 \\ & 3x_1 + 5x_2 + 10x_3 + 15x_4 \leqslant 100 \\ & x_j \geqslant 0, \; j = 1, 2, 3, 4. \end{aligned}$$

and

$$\begin{aligned} \text{Max} \quad & Z = 4x_1 + 5x_2 + 9x_3 + 11x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \leqslant 20 \\ & 7x_1 + 5x_2 + 3x_3 + 2x_4 \leqslant 120 \\ & 3x_1 + 5x_2 + 10x_3 + 15x_4 \leqslant 130 \\ & x_j \geqslant 0, \; j = 1, 2, 3, 4. \end{aligned}$$

The optimum value of the objective function of these LPPs are respectively $z_o = 99.29$ and $z_1 = 0$ The membership functions of the objective function and the constraints are as follows.

$$\mu_0(x) = \begin{cases} 1 & \text{for} \quad cx \geq 130 \\ (cx - 99.29)/30.71 & \text{for } 99.29 < cx < 130 \\ 0 & \text{for} \quad cx \leq 99.29 \end{cases}$$

$$\mu_1(x) = \begin{cases} 1 & \text{for} \quad g_1(x) \leq 15 \\ (20 - g_1)/5 & \text{for } 15 < g_1(x) < 20 \\ 0 & \text{for} \quad g_1 \geq 20 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{for} \quad g_1(x) \leq 80 \\ (120 - g_2)/40 & \text{for } 80 < g_2(x) < 120 \\ 0 & \text{for} \quad g_2 \geq 120 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{for } g_1(x) \leq 100 \\ (130 - g_3)/30 & \text{for } 100 < g_3(x) < 130 \\ 0 & \text{for } g_3 \geq 130 \end{cases}$$

Using Werners' method the crisp LPP equivalent to the given fuzzy LPP is

$$\begin{aligned} \text{Maximize} \quad & z = \alpha \\ \text{subject to} \quad & \mu_0(x) \geq \alpha \\ & \mu_1(x) \geq \alpha \\ & \mu_2(x) \geq \alpha \\ & \mu_3(x) \geq \alpha \\ & 0 \leq \alpha \leq 1 \\ & x \geqslant 0, \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & z = \alpha \\ \text{s.t.} \quad & 4x_1 + 5x_2 + 9x_3 + 11x_4 - 30.71\alpha \leqslant 99.29 \\ & x_1 + x_2 + x_3 + x_4 + 5\alpha \leqslant 20 \\ & 7x_1 + 5x_2 + 3x_3 + 2x_4 + 40\alpha \leqslant 120 \\ & 3x_1 + 5x_2 + 10x_3 + 15x_4 + 30\alpha \leqslant 130 \\ & 0x_1 + 0x_2 + 0x_3 + 0x_4 + \alpha \leqslant 1 \\ & x_j \geqslant 0, \ j = 1, 2, 3, 4. \\ & \alpha \geq 0 \end{aligned}$$

The optimum solution is obtained as $x_1 = 8.57$, $x_2 = 0$, $x_3 = 8.93$, $x_4 = 0$
$Z_{max} = 114.64$,$\alpha = 0.5$.

Actual used resources are found as
17.5,86.78 and 115.01 respectively.

3.6 Zimmermann's method to solve fuzzy LPP

The general model of a LPP with fuzzy objective and fuzzy constraints is given by

$$\begin{aligned} \widetilde{max} \quad & z = cx \\ \text{subject to} \quad & (Ax)_i \lessapprox b_i, i = 1, 2, \ldots m \\ & x \geqslant 0, \end{aligned}$$

The fuzzy constraint $(Ax)_i \lessapprox b_i$. for each $i = 1, 2, \ldots m$ has the meaning that if $(Ax)_i \leq b_i$ then the i^{th} constraint is absolutely satisfied, if $(Ax)_i \geq b_i + p_i$ then the i^{th}constraint is absolutely violated, where p_i is the maximum tolerance from b_i. If $b_i < (Ax)_i < b_i + p_i$ then the i^{th} constraint is satisfied partially, For$(Ax)_i \in (b_i, b_i + p_i)$, the membership function is monotonically decreasing as a linear function. The membership function is defined for each $i = 1, 2, \ldots, n$, as

$$\mu_i(x) = \begin{cases} 1 & \text{for } (Ax)_i \le b_i \\ \{b_i + p_i - (Ax)_i\}/p_i & \text{for } \ b_i < (Ax)_i < b_i + p_i \\ 0 & \text{for } (Ax)_i > b_i + p_i \end{cases}$$

The fuzzifier $\widetilde{max}$ is understood in the sense of the satisfaction of an aspiration levels z_0 as best as possible, Let p_0 be the permissible tolerance for the objective function. The membership function$\mu_0(x)$ for the objective function is taken to be nondecreasing and continuous and is defined as

$$\mu_0(x) = \begin{cases} 1 & \text{for } \ cx \ge z_0 \\ (cx + p_0 - z_0)/p_i & \text{for } z_0 - p_0 < cx < z_0 \\ 0 & \text{for } \ \ cx \le z_0 - p_0 \end{cases}$$

To identify the fuzzy decision Zimmermann employed Bellman and Zade .This leads . to the following crisp LPP

$$\begin{aligned} \text{Maximize} \quad & z = \alpha \\ \text{subject to} \quad & \mu_0(x) \ge \ \alpha \\ & \mu_i(x) \ge \ \alpha, i = 1, 2, \ldots, m \\ & x \geqslant 0, \\ & 0 \le \alpha \le 1, \end{aligned}$$

or,

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & cx \ge \ z_0 - (1 - \alpha)p_0 \\ & (Ax)_i \le \ b_i + (1 - \alpha)p_i, i = 1, 2, \ldots, m \\ & x \geqslant 0, \\ & 0 \le \alpha \le 1, \end{aligned}$$

We note here that if (x^*, α^*) is, the optimal solution of this equivalent crips LPP then$\dot{\alpha}$ is the degree upto which the aspiration level z_0 of the decision maker is met.
In expain Zimmermann method for solving fuzzy LPP Zimmermann consider the following example.

3.6.1 **Example**. Zirnmennann considered the example

$$\begin{aligned} \widetilde{Max} \quad & Z = x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + 3x_2 \lessapprox 21 \\ & x_1 \quad + 3x_2 \lessapprox 27 \\ & 4x_1 \quad + 3x_2 \lessapprox 45 \\ & 3x_1 \quad + x_2 \lessapprox 30 \\ & x_j \geqslant 0, \ j = 1, 2, 3, 4. \end{aligned}$$

The aspiration level z_o and tolerance levels p_i are taken as $z_o = 14.5$, $p_o = 2$, $p_1 = 3$, $p_2 = 6$ and $p_3 = 6$. Using Zimmermann's method the crisp LPP equivalent to this fuzzy LPP is given by

$$\begin{aligned}
\text{Maximize} \quad & Z = \alpha \\
\text{s.t.} \quad & x_1 + x_2 \;\geq\; 14.5 - 2(1-\alpha) \\
& -x_1 + 3x_2 \;\leq\; 21 + 3(1-\alpha) \\
& x_1 + 3x_2 \;\leq\; 27 + 6(1-\alpha) \\
& 4x_1 + 3x_2 \;\leq\; 45 + 6(1-\alpha) \\
& 3x_1 + x_2 \;\leq\; 30 \\
& \alpha \leq 1, \\
& x_j \geqslant 0,\; j = 1,2. \;\; \alpha \geq 0
\end{aligned}$$

or,

$$\begin{aligned}
\text{Maximize} \quad & Z = \alpha \\
\text{s.t.} \quad & 2\alpha - x_1 - x_2 \;\geq\; -12.5 \\
& 3\alpha - x_1 + 3x_2 \;\leq\; 24 \\
& 6\alpha + x_1 + 3x_2 \;\leq\; 33 \\
& 6\alpha + 4x_1 + 3x_2 \;\leq\; 51 \\
& 3x_1 + x_2 \;\leq\; 30 \\
& \alpha \leq 1, \\
& x_j \geqslant 0,\; j = 1,2. \;\; \alpha \geq 0
\end{aligned}$$

Using simplex method the optimal solution is obtained as $x_1^* = 6$, $x_2^* = 7.75$,$z_{max} = 13.75$ and $\alpha^* = 0.625$.

3.7 Illustrative Examples

3.7.1 Example.

Using Verdegay's method solve the fuzzy LPP considered in example 78.4.1 **Solution.** The fuzzy LPP is

$$\begin{aligned}
\text{Maximize} \quad & Z = 2x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 0x_2 \;\leq\; 3 \;\; or \;\; 4 \\
& x_1 + x_2 \;\leq\; 4 \;\; or \;\; 6 \\
& 0.5x_1 + x_2 \;\leq\; 3 \;\; or \;\; 5 \\
& x_j \geqslant 0,\; j = 1,2.
\end{aligned}$$

The membership functions of the constraints are given by

$$\mu_1(x_1, x_2) = \begin{cases} 1 & for \; x_1 \leq 3 \\ (4 - x_1) & for \; 3 < x_1 < 4 \\ 0 & for \; x_1 \geq 4 \end{cases}$$

$$\mu_2(x_1,x_2)=\begin{cases}1 & \text{for } x_1+x_2\le 4\\ (6-x_1-x_2)/2 & \text{for } 4<x_1+x_2<6\\ 0 & \text{for } x_1+x_2\ge 6\end{cases}$$

$$\mu_3(x_1,x_2)=\begin{cases}1 & \text{for } 0.5x_1+x_2\le 3\\ (5-0.5x_1-x_2)/2 & \text{for } 3<0.5x_1+x_2<5\\ 0 & \text{for } 0.5x_1+x_2\ge 5\end{cases}$$

Using Verdegay's method the crisp parametric programming problem equivalent to the given fuzzy LPP is given by

$$\begin{aligned}\text{Maximize}\quad & z=2x_1+x_2\\ \text{subject to}\quad & \mu_0(x)\ge\ \alpha\\ & \mu_1(x)\ge\ \alpha\\ & \mu_2(x)\ge\ \alpha\\ & \mu_3(x)\ge\ \alpha\\ & \alpha\ge 0\\ & x,\alpha\geqslant 0,\qquad where\ \ x=(x_1,x_2)\end{aligned}$$

$$\begin{aligned}\text{Maximize}\quad & z=2x_1+x_2\\ \text{s.t.}\quad & x_1\ \le\ 3+(1-\alpha)\\ & x_1+x_2\ \le\ 4+(1-\alpha)2\\ & 0.5x_1+x_2\ \le\ 3+(1-\alpha)2\\ & \alpha\le 1\\ & \alpha,x_j\geqslant 0,\ j=1,2.\end{aligned}$$

Let$\theta=1-\alpha$. since $0\le\alpha\le 1$ we have $0\le\theta\le 1$
The Lpp becomes

$$\begin{aligned}\text{Maximize}\quad & z=2x_1+x_2\\ \text{s.t.}\quad & x_1\ \le\ 3+\theta\\ & x_1+x_2\ \le\ 4+2\theta\\ & 0.5x_1+x_2\ \le\ 3+2\theta\\ & \theta\ge 0\\ & \theta\le 1\\ & x_j\geqslant 0,\ j=1,2.\end{aligned}$$

or,

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 \ \leq \ 3 + \theta \\ & x_1 + x_2 \ \leq \ 4 + 2\theta \\ & x_1 + 2x_2 \ \leq \ 6 + 4\theta \\ & 0 \leq \theta \leq 1 \\ & x_j \geqslant 0, \ j = 1, 2. \end{aligned}$$

To solve this parametric LPP we first solve the LPP taking $\theta = 0$ i.e. we solve the following

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 \ \leq \ 3 \\ & x_1 + x_2 \ \leq \ 4 \\ & x_1 + 2x_2 \ \leq \ 6 \\ & x_j \geqslant 0, \ j = 1, 2. \end{aligned}$$

Introducing slack variables x_3, x_4,x_5 we get

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{s.t.} \quad & x_1 + 0x_2 + 0x_3 \ = \ 3 \\ & x_1 + x_2 + x_4 \ = \ 4 \\ & x_1 + 2x_2 + x_5 \ = \ 6 \\ & x_j \geqslant 0, \ j = 1, 2, 3, 4, 5. \end{aligned}$$

		c_j	2	1	0	0	0	
c_B	x_B	b	y_1	y_2	y_3	y_4	y_5	min ratio
0	y_3	3	$\boxed{1}$	0	1	0	0	3
0	y_4	4	1	1	0	1	0	4
0	y_5	6	1	2	0	0	1	6
	$z = 0$	$z_j - c_j$	-2	-1	0	0	0	
2	y_1	3	1	0	1	0	0	-
0	y_4	1	0	$\boxed{1}$	-1	1	0	1
0	y_5	3	0	2	-1	0	1	3/2
	$z = 6$	$z_j - c_j$	0	-1	2	0	0	
2	y_1	3	1	0	1	0	0	
1	y_2	1	0	1	-1	1	0	
0	y_5	1	0	0	1	-2	1	
	$z = 7$	$z_j - c_j$	0	0	1	1	0	

From this final table we get the optimal values of the basic variables for the parametric LPP as $x_B = b + \theta y_3 + 2\theta y_4 + 4\theta y_5$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + \theta \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 2\theta \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + 4\theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3+\theta \\ 1-\theta+2\theta \\ 1+\theta-4\theta+4\theta \end{bmatrix}$$

or,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3+\theta \\ 1\theta \\ 1+\theta \end{bmatrix}$$

The optimal value of the objective function is given by

$$\begin{aligned} \dot{Z} &= Z + \theta(Z_3 - C_3) + 2\theta(Z_4 - C_4) + 4\theta(Z_5 - C_5) \\ &= 7 + \theta \cdot 1 + 2\theta \cdot 1 + 4\theta \cdot 0 \\ &= 7 + 3\theta. \end{aligned} \tag{3.3}$$

Hence the optimal solution of the parametric LPP is $x_1 = 3+\theta$
$x_2 = 1+\theta$
and $z_{max} = 7 + 3\theta$ where $0 \le \theta \le 1$.

3.7.2 **Example**

. Using Werners' method solve the fuzzy LPP considered in Example 78.4.1. **Solution**. The fuzzy LPP is

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 0x_2 \ \le \ 3 \ \ \textit{or} \ \ 4 \\ & x_1 + x_2 \ \le \ 4 \ \ \textit{or} \ \ 6 \\ & 0.5x_1 + x_2 \ \le \ 3 \ \ \textit{or} \ \ 5 \\ & x_j \geqslant 0, \ j = 1, 2. \end{aligned}$$

In the Wemers' method the membership function of the objective function is found with the help of optimal values of the objective function of the following two LPP.

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 0x_2 \ \le \ 3 \\ & x_1 + x_2 \ \le \ 4 \\ & 0.5x_1 + x_2 \ \le \ 3 \\ & x_j \geqslant 0, \ j = 1, 2. \end{aligned}$$

and

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 0x_2 \ \leq \ 4 \\ & x_1 + x_2 \ \leq \ 6 \\ & 0.5x_1 + x_2 \ \leq \ 5 \\ & x_j \geqslant 0, \ j = 1, 2. \end{aligned}$$

The optimal solution of the first LPP is $x_1 = 3$, $x_2 = 1$ and maximum value of z is 7 $\therefore$ $z_o = 7$. The optimal solution of the second LPP is $x_1 = 4$, $x_2 = 2$ and the maximum value of z is 10 $\therefore$ $z_1 = 10$. The membership function of the objective function is given by

$$\mu_0(x) = \begin{cases} 1 & for\ 2x_1 + x_2 \geq 10 \\ (2x_1 + x_2 - 7)/3 & for\ 7 < 2x_1 + x_2 < 10 \\ 0 & for\ 2x_1 + x_2 \leq 7 \end{cases}$$

The membership function of the constraints are given by

$$\mu_1(x_1, x_2) = \begin{cases} 1 & for\ x_1 \leq 3 \\ (4 - x_1) & for\ 3 < x_1 < 4 \\ 0 & for\ x_1 \geq 4 \end{cases}$$

$$\mu_2(x_1, x_2) = \begin{cases} 1 & for\ x_1 + x_2 \leq 4 \\ (6 - x_1 - x_2)/2 & for\ 4 < x_1 + x_2 < 6 \\ 0 & for\ x_1 + x_2 \geq 6 \end{cases}$$

$$\mu_3(x_1, x_2) = \begin{cases} 1 & for\ 0.5x_1 + x_2 \leq 3 \\ (5 - 0.5x_1 - x_2)/2 & for\ 3 < 0.5x_1 + x_2 < 5 \\ 0 & for\ 0.5x_1 + x_2 \geq 5 \end{cases}$$

Using Wemers' method the crisp LPP equivalent to the given fuzzy LPP is given by

$$\begin{aligned} \text{Maximize} \quad & z = cx \\ \text{subject to} \quad & cx \geq \ z_1 - (1 - \alpha)(z_1 - z_0) \\ & (Ax)_i \geq \ b_i + (1 - \alpha)p_i \\ & x \geqslant 0, \\ & 0 \leq \alpha \leq 1, \end{aligned}$$

or,

$$\begin{aligned} \text{Maximize} \quad & z = \alpha \\ \text{subject to} \quad & 2x_1 + x_2 \;\geq\; 10 - (1-\alpha)3 \\ & x_1 \;\leq\; 3 + (1-\alpha)\cdot 1 \\ & x_1 + x_2 \;\leq\; 4 + (1-\alpha)2 \\ & 0.5x_1 + x_2 \;\leq\; 3 + (1-\alpha)2 \\ & x_1, x_2 \geq 0, \\ & 0 \leq \alpha \leq 1, \end{aligned}$$

or,

$$\begin{aligned} \text{Maximize} \quad & z = \alpha \\ \text{subject to} \quad & 2x_1 + x_2 + 3\alpha \;\geq\; 7 \\ & x_1 + 0x_2 + \alpha \;\geq\; 4 \\ & x_1 + x_2 + 2\alpha \;\leq\; 6 \\ & 0.5x_1 + x_2 + 4\alpha \;\leq\; 10 \\ & \alpha \leq 1, \\ & \alpha, x_1, x_2 \geq 0, \end{aligned}$$

Solution of this crisp LPP gives the optimal solution of the given fuzzy LPP.

3.7.3 **Example.**

Using Zimmermann's method solve the fuzzy LPP considered in the example 78.4.1
Solution. The fuzzy LPP is

$$\begin{aligned} \widetilde{Maximize} \quad & Z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 0x_2 \;\leq\; 3 \;\; or \;\; 4 \\ & x_1 + x_2 \;\leq\; 4 \;\; or \;\; 6 \\ & 0.5x_1 + x_2 \;\leq\; 3 \;\; or \;\; 5 \\ & x_j \geqslant 0,\; j = 1, 2. \end{aligned}$$

Let us take here the aspiration level of the objective function value as 12 and the permissible tolerance of it as 3. According to Zimmermann's method the membership function of the objective function is given by

$$\mu_0(x) = \begin{cases} 1 & for\ 2x_1 + x_2 \geq 12 \\ (2x_1 + x_2 - 9)/3 & for\ 9 < 2x_1 + x_2 < 12 \\ 0 & for\ 2x_1 + x_2 \leq 9 \end{cases}$$

The membership function of the constraints are given by

$$\mu_1(x_1,x_2)=\begin{cases}1 & for\ x_1\le 3\\ (4-x_1) & for\ 3<x_1<4\\ 0 & for\ x_1\ge 4\end{cases}$$

$$\mu_2(x_1,x_2)=\begin{cases}1 & for\ x_1+x_2\le 4\\ (6-x_1-x_2)/2 & for\ 4<x_1+x_2<6\\ 0 & for\ x_1+x_2\ge 6\end{cases}$$

$$\mu_3(x_1,x_2)=\begin{cases}1 & for\ 0.5x_1+x_2\le 3\\ (5-0.5x_1-x_2)/2 & for\ 3<0.5x_1+x_2<5\\ 0 & for\ 0.5x_1+x_2\ge 5\end{cases}$$

Using Zimmermann's method the crisp LPP equivalent to the given fuzzy LPP is given by

$$\begin{aligned}\text{Maximize}\quad & z=\alpha\\ \text{subject to}\quad & \mu_0(x_1,x_2)\ge\ \alpha\\ & \mu_i(x_1,x_2)\ge\ \alpha, i=1,2,3\\ & \alpha,x_1,x_2\geqslant 0,\\ & \alpha\le 1,\end{aligned}$$

or,

$$\begin{aligned}\text{Maximize}\quad & z=\alpha\\ \text{subject to}\quad & 2x_1+x_2\ \ge\ 12-(1-\alpha)3\\ & x_1\ \le\ 3+(1-\alpha)\cdot 1\\ & x_1+x_2\ \le\ 4+(1-\alpha)\cdot 2\\ & 0.5x_1+x_2\ \le\ 3+(1-\alpha)2\\ & \alpha,x_1,x_2\ge 0,\\ & \alpha\le 1,\end{aligned}$$

or,

$$\begin{aligned}\text{Maximize}\quad & z=\alpha\\ \text{subject to}\quad & 2x_1+x_2+3\alpha\ \ge\ 9\\ & x_1+0x_2+\alpha\ \le\ 4\\ & x_1+x_2+2\alpha\ \le\ 6\\ & 0.5x_1+x_2+4\alpha\ \le\ 5\\ & \alpha\le 1,\\ & \alpha,x_1,x_2\ge 0,\end{aligned}$$

Using simplex method we get the optimal solution of this crisp LPP and that is the optimal solution of the given fuzzy LPP.

3.8 Linear Programming Problems with Fuzzy Parameters

Definition: The partial ordering $\leq$ for triangular fuzzy number is defined by $(al,\, am,\, ar) \leq (bl,\, bm,\, br)$ iff $al \leq bl$, $am \leq bm$ and $ar \leq br$.
Theorem: *(Pareto Optimality)* For a problem with p objective functions, the point $x^* \in X$ is a Pareto Optimal solution if there does not exist $x \in X$ such that if $Z_i(x) \geq Z_i(x^*)$ for all i and $Z_j(x) > Z_j(x^*)$ for at least one j.
The Linear Programming problem with fuzzy parameters are given as

$$\begin{aligned} &\text{Maximize } \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j\, x_j \\ &\text{Subject to the constraints} \\ &\quad \sum_{j=1}^{n} \tilde{a}_{ij}\, x_j \leq \tilde{b}_i,\ i = 1,\, 2,\, ...,\, m. \\ &\quad x_j \geq 0,\ j = 1,\, 2, ..., n. \end{aligned} \tag{3.4}$$

where $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy numbers.
Considering the fuzzy parameters as triangular fuzzy numbers, we have

$$\begin{aligned} &\text{Maximize } \tilde{Z} = \sum_{j=1}^{n} (cl,\, cm,\, cr)_j\, x_j \\ &\text{Subject to the constraints} \\ &\quad \sum_{j=1}^{n} (al,\, am,\, ar)_{ij}\, x_j \leq (bl,\, bm,\, br)_i,\ i = 1,\, 2,\, ...,\, m. \\ &\quad x_j \geq 0,\ j = 1,\, 2, ..., n. \end{aligned} \tag{3.5}$$

where $(cl,\, cm,\, cr)_j$ is the j-th fuzzy coefficient in the objective function. $(al,\, am,\, ar)_{ij}$ is the fuzzy coefficient of j-th variable in the i-th constraints and $(bl,\, bm,\, br)_i$ is the i-th fuzzy resource. These problem can be solved by converting into equivalent crisp multi-objective linear problem as

$$\begin{aligned} &\text{Maximize } (Z_1,\, Z_2,\, Z_3) \\ &\text{where } Z_1 = \sum_{j=1}^{n} cl_j\, x_j,\ Z_2 = \sum_{j=1}^{n} cm_j\, x_j,\ Z_3 = \sum_{j=1}^{n} cr_j\, x_j. \\ &\text{Subject to the constraints} \\ &\quad \sum_{j=1}^{n} al_{ij}\, x_j \leq bl_i,\ i = 1,\, 2,\, ...,\, m. \\ &\quad \sum_{j=1}^{n} am_{ij}\, x_j \leq bm_i,\ i = 1,\, 2,\, ...,\, m. \\ &\quad \sum_{j=1}^{n} ar_{ij}\, x_j \leq br_i,\ i = 1,\, 2,\, ...,\, m. \\ &\quad x_j \geq 0,\ j = 1,\, 2, ..., n. \end{aligned} \tag{3.6}$$

The above multi-objective optimization problem can be solved by weighted objective function. The reduced problem is as

$$\begin{aligned}
&\text{Maximize } w_1 Z_1 + w_2 Z_2 + w_3 Z_3 \\
&\text{where } Z_1 = \sum_{j=1}^{n} cl_j\, x_j,\ Z_2 = \sum_{j=1}^{n} cm_j\, x_j,\ Z_3 = \sum_{j=1}^{n} cr_j\, x_j. \\
&\text{Subject to the constraints} \\
&\quad w_1 + w_2 + w_3 = 1 \\
&\quad \sum_{j=1}^{n} al_{ij}\, x_j \le bl_i,\ i = 1,\ 2,\ ..., m. \\
&\quad \sum_{j=1}^{n} am_{ij}\, x_j \le bm_i,\ i = 1,\ 2,\ ..., m. \\
&\quad \sum_{j=1}^{n} ar_{ij}\, x_j \le br_i,\ i = 1,\ 2,\ ..., m. \\
&\quad x_j \ge 0,\ j = 1,\ 2, ..., n.
\end{aligned} \tag{3.7}$$

Solution can be obtained by solving the problem with different weights.

3.9 Multi-objective fuzzy linear programming Problem

The multi-objective fuzzy linear programming problem is given as

$$\begin{aligned}
&\text{Maximize } \tilde{Z}^1,\ \tilde{Z}^2,\ ...,\ \tilde{Z}^p \\
&\text{where } \tilde{Z}^k = \sum_{j=1}^{n} \tilde{c}_j^k\, x_j,\ k = 1,\ 2,\ ..., p. \\
&\text{Subject to the constraints} \\
&\quad \sum_{j=1}^{n} \tilde{a}_{ij}\, x_j \le \tilde{b}_i,\ i = 1,\ 2,\ ..., m. \\
&\quad x_j \ge 0,\ j = 1,\ 2, ..., n.
\end{aligned} \tag{3.8}$$

where $\tilde{c}_j^k$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy numbers.
Considering the objective function coefficient ($\tilde{c}_j^k$), technological coefficient ($\tilde{a}_{ij}$) and resource coefficient ($\tilde{b}_i$) as triangular fuzzy numbers. The above problem can be represented as

$$\begin{aligned}
&\text{Maximize } \tilde{Z}^1,\ \tilde{Z}^2,\ ...,\ \tilde{Z}^p \\
&\text{where } \tilde{Z}^k = \sum_{j=1}^{n} (cl,\ cm,\ cr)_j^k\, x_j,\ k = 1,\ 2,\ ..., p. \\
&\text{Subject to the constraints} \\
&\quad \sum_{j=1}^{n} (al,\ am,\ ar)_{ij}\, x_j \le (bl,\ bm,\ br)_i,\ i = 1,\ 2,\ ..., m. \\
&\quad x_j \ge 0,\ j = 1,\ 2, ..., n.
\end{aligned} \tag{3.9}$$

where $(cl,\ cm,\ cr)_j^k$ is the j-th fuzzy coefficient in the k-th objective function. $(al,\ am,\ ar)_{ij}$ is the fuzzy coefficient of j-th variable in the i-th constraints and $(bl,\ bm,\ br)_i$ is the i-th fuzzy resource. Since the coefficient of objective function is represented by triangular fuzzy numbers. Therefore, each objective function gives rise to 3 crisp objective functions. Hence the converted problem will involve 3p objective functions to be optimized. The converted problem becomes

$$\begin{aligned}
&\text{Maximize } (Z_1^1,\, Z_2^1,\, Z_3^1,\, Z_1^2,\, Z_2^2,\, Z_3^2,\, ..., Z_1^p,\, Z_2^p,\, Z_3^p,)\\
&\text{where } Z_1^k = \sum_{j=1}^{n} cl_j^k\, x_j,\; Z_2^k = \sum_{j=1}^{n} cm_j^k\, x_j,\; Z_3^k = \sum_{j=1}^{n} cr_j^k\, x_j,\; k = 1, 2, ..., p.\\
&\text{Subject to the constraints}\\
&\quad \sum_{j=1}^{n} al_{ij}\, x_j \le bl_i,\; i = 1, 2, ..., m.\\
&\quad \sum_{j=1}^{n} am_{ij}\, x_j \le bm_i,\; i = 1, 2, ..., m.\\
&\quad \sum_{j=1}^{n} ar_{ij}\, x_j \le br_i,\; i = 1, 2, ..., m.\\
&\quad x_j \ge 0,\; j = 1, 2, ..., n.
\end{aligned} \tag{3.10}$$

The weighted objective function of the above problem using pareto method can be represented as

$$\begin{aligned}
&\text{Maximize } w_{11} Z_1^1 + w_{12}\, Z_2^1 + w_{13}\, Z_3^1 + w_{21}\, Z_1^2 + ... + w_{p2}\, Z_2^p + w_{p3}\, Z_3^p,)\\
&\text{where } Z_1^k = \sum_{j=1}^{n} cl_j^k\, x_j,\; Z_2^k = \sum_{j=1}^{n} cm_j^k\, x_j,\; Z_3^k = \sum_{j=1}^{n} cr_j^k\, x_j,\; k = 1, 2, ..., p.\\
&\text{Subject to the constraints}\\
&\quad w_{k1} + w_{k2} + w_{k3} = 1,\; k = 1, 2, ..., p.\\
&\quad \sum_{j=1}^{n} al_{ij}\, x_j \le bl_i,\; i = 1, 2, ..., m.\\
&\quad \sum_{j=1}^{n} am_{ij}\, x_j \le bm_i,\; i = 1, 2, ..., m.\\
&\quad \sum_{j=1}^{n} ar_{ij}\, x_j \le br_i,\; i = 1, 2, ..., m.\\
&\quad x_j \ge 0,\; j = 1, 2, ..., n.
\end{aligned} \tag{3.11}$$

Solution can be obtained by solving the problem with different weights.
Ex.

$$\begin{aligned}
&\text{Maximize } \tilde{Z}^1 = (7,\, 10,\, 14)x_1 + (20,\, 25,\, 35)x_2,\; \tilde{Z}^2 = (10,\, 14,\, 25)x_1 + (25,\, 35,\, 40)x_2\\
&\text{Subject to the constraints}\\
&\quad (1,\, 3,\, 4)x_1 + (2,\, 6,\, 7)x_2 \le (8,\, 13,\, 15)\\
&\quad (3,\, 4,\, 6)x_1 + (1,\, 6,\, 10)x_2 \le (3,\, 7,\, 9)\\
&\quad x_1,\, x_2 \ge 0.
\end{aligned} \tag{3.12}$$

3.10 Summary

In this module we have discussed applications of the fuzzy set theory developed in the earlier modules. Applicatons area arc confined here mainly to fuzzy linear programming, Classification of fuzzy LPP is discussed. The pioneering work of Bellman and Zadeh for getting decision of fuzzy environment is considered for solving fuzzy LPP. Different methods developed by verdegay, Wemers and Zimmermann are discussed in details with examples to explain the methods.