M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Semester-IV

Paper-MTM402 Unit-1

Fuzzy Mathematics with Applications

(Interval, fuzzy sets, fuzzy number and their arithmetic)

Unit Structure:

2.1 Introduction

In this module we first consider the definition and arithmetic of Intervals. The notion of Interval arithmetic is then used to develop the arithmetic of Fuzzy numbers. Fuzzy numbers are nothing but particular fuzzy sets. Hence operations on fuzzy sets are discussed first. Then the notion of Interval arithmetic and operations on fuzzy sets are used for the development of fuzzy numbers arithmetic.

2.2 Interval Numbers

2.2.1 Definition

An interval number is defined as an ordered pair of finite real numbers $[a,\ b]$ where $a \leq b$. When $a = b$ the interval number $[a,\ b]$ degenerates to the scalar real number 'a'.

An interval number can be thought as

(i) an extension of the concept of a real number and also as a subset of the real line [Moore 1979, Alefeld & Herzberger (1983)].

(ii) a simplest form of tolerance-type uncertainty with no information about the probabilities within this tolerance range (Nauyen & Kreinovich, 2005).

(iii) a grey number whose exact value is unknown but a range within which the value lies is known [Liu & Lin, 1998].

Thus an interval number represents a set of possible values that a particular entity or variable may assume without any prior assumption about exact value and probability measure. In other words, interval numbers should be used whenever decision variables can assume different values, but a probability measure on these values is not available or justifiable. In reality, inexactness of this kind occurs in countless numbers. An interval number may also be called as an interval.

2.2.2 Set Operations on Intervals

Definition

(i) Equality: Two intervals [a, b] and [c, d] are said to be equal if and only if a = c and b = d.

(ii) Intersection: The intersection of two intervals [a, b] and [c, d] is defined as

$$[a,\ b] \cap [c,\ d] = [\ max\{a, c\}, min\{b, d\}\ ]$$

Note: $[a,\ b] \cap [c,\ d] = \phi$ if and only if $a > d$ or $c > b$.

(iii) Union: The union of two intervals $[a,\ b]$ and $[c,\ d]$ is defined as

$$[a, b] \cup [c, d] = [\ min\{a, c\}, max\{b, d\}\ ]$$

provided that $[a, b] \cap [c, d] \neq \phi$

(iv) Inclusion: The interval $[a,\ b]$ is said to be included in $[c,\ d]$ if and only if both $c < a$ and $b < d$. It is written as $[a, b] \subset [c, d]$

For given two intervals $I_1 = [a,\ b]$ and $I_2 = [c,\ d]$ the following six cases may arise:
(i) $a > d$ (ii) $c > b$ (iii) $a > c$ and $b < d$ (iv) $c > a$ and $d < b$
(v) $a < c < b < d$ and (vi) $c < a < d < b$.

Tabl 2.1 shows the various combinations of set-theoretic intersection and set-theoretic union for these six possible combinations of a, b, c and d.

Table 2.1: Various combinations of set-theoretic intersection and union.

Cases	Intersection ($\cap$)	Union ($\cup$)
(i) $a > d$	ϕ	$[c, d] \cup [a, b]$
(ii) $c > b$	ϕ	$[a, b] \cup [c, d]$
(iii) $a > c, b < d$	$[a, b]$	$[c, d]$
(iv) $c > a, d < b$	$[c, d]$	$[a, b]$
(v) $a < c < b < d$	$[c, b]$	$[a, d]$
(vi) $c < a < d < b$	$[a, d]$	$[c, b]$

2.2.3 Interval Arithmetic

Let $[a_1, b_1], [a_2, b_2]$ and $[a,\ b]$ be intervals. Then addition, subtraction, multiplication and division are defined as follows:

(i) Addition :

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$$

(ii) Subtraction:

$$[a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2]$$

(iii) Multiplication:

$$[a_1, b_1] \times [a_2, b_2] = [\ min\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}, max\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}\ ]$$

(iv) Division:

$$[a_1, b_1]/[a_2, b_2] = [\ min\{a_1/a_2, a_1/b_2, b_1/a_2, b_1/b_2\}, max\{a_1/a_2, a_1/b_2, b_1/a_2, b_1/b_2\}\ ]$$

provided that $0 \notin [a_2, b_2]$

(v) Scalar Multiplication :

$$\begin{aligned} k[a, b] &= [ka, kb] \text{ for } k \geq 0 \\ &= [kb, ka] \text{ for } k < 0 \end{aligned}$$

(vi) Reciprocal:

$$\text{If } 0 \notin [a, b\ ] \text{ then } [a, b]^{-1} = [\ min\ \{1/a, 1/b\}, max\ \{1/a, 1/b\}\ ]$$

If $0 \in [a, b\ ]$ then $[a, b]^{-1}$ is undefined.

For non-negative intervals multiplication, division and reciprocal reduces to the following.

Multiplication :

$$[a_1, b_1] \times [a_2, b_2] = [a_1a_2, b_1b_2]$$

Division:

$$[a_1, b_1]/[a_2, b_2] = [a_1/b_2, b_1/a_2]$$

Inverse:

$$[a, b]^{-1} = [1/b, 1/a]$$

Remarks :

From $[3, 14] + [5, 20] = [8, 34]$ we note that for any $x \in [3, 14]$ and any $y \in [5, 20]$, it is guaranteed that $x + y \in [8, 34]$. Also from $[2, 8] - [3, 10] = [-8, 5]$ we note that for any $x \in [2, 8]$ and for any $y \in [3, 10]$, it is guaranteed that $x - y \in [-8, 5]$. So interval arithmetic intends to obtain an interval as the result of an operation such that the resulting interval contains all possible solutions.

Again interval arithmetic may produce some unusual results that could seem to be inconsistent with the ordinary numerical solutions. As for example ordinary results gives $[2, 6] - [2, 6] = [0, 0]$, but from interval arithmetic we have $[2, 6] - [2, 6] = [-4, 4]$ and not $[0, 0]$. Here we note that $[0, 0] \in [-4, 4]$ i.e. $[-4, 4]$ contains 0 but not only 0, many others also i.e. 0 as well as all other possible solutions.

2.2.4 Algebraic Properties of Interval Arithmetic

We can easily prove the following properties of interval arithmetic.

Let X, Y, Z be intervals, then we have

i) $X + Y = Y + X$

ii) $(X + Y) + Z = X + (Y + Z)$

iii) $(XY)Z = X(YZ)$

iv) $XY = YX$

v) $Z + 0 = 0 + Z = Z$ and $Z0 = 0Z = 0$ where $0 = [0, 0]$

vi) $ZI = IZ = Z$ where $I = [1, 1]$

vii) $Z(X + Y) \neq ZX + ZY$, except when

   (a) $Z = [z, z]$ is a point or
   (b) $X = Y = 0$ or
   (c) $xy \geq 0$ for all $x \in X$ and $y \in Y$.
   In general, only the subdistributive law holds :

$$Z(X + Y) \subseteq ZX + ZY.$$

2.3 Operations on Fuzzy Sets

Now we proceed to define certain standard set theoretic operations for fuzzy sets.

2.3.1 Definition: Empty fuzzy set

A fuzzy set $\tilde{A}$ defined over the universe X is said to be empty if its membership function is identically zero i.e. if $\mu_{\tilde{A}}(x) = 0$ for all $x \in X$.

2.3.2 Definition: Subset

A fuzzy set $\tilde{A}$ is said to be a subset of a fuzzy set $\tilde{B}$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ for all $x \in X$. This is denoted by $\tilde{A} \subseteq \tilde{B}$.

2.3.3 Definition: Equality of fuzzy sets

Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are said to be equal if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$ i.e. if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ for all $x \in X$.

2.3.4 Definition: Complement

The complement of a fuzzy set $\tilde{A}$ defined over the universal set X is another fuzzy $\tilde{A}'$ defined by the membership function

$$\mu_{\tilde{A}'}(x) = 1 - \mu_{\tilde{A}}(x) \text{ for all } x \in X$$

2.3.5 Definition: Union

The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is another fuzzy set $\tilde{C}$ defined by the membership function

$$\mu_{\tilde{C}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \text{ for all } x \in X.$$

2.3.6 Definition : Intersection

The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is another fuzzy set $\tilde{C}$ defined by the membership function

$$\mu_{\tilde{C}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \text{ for all } x \in X.$$

Before studying the properties of fuzzy sets we state the standard properties of crisp sets.

2.3.7 Properties of Crisp Sets

The following are the important properties of crisp sets.
i) Commutativity: $A \cup B = B \cup A$ & $A \cap B = B \cap A$
ii) Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$ & $(A \cap B) \cap C = A \cap (B \cap C)$
iii) Distributive laws : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
iv) De Morgan's laws: $(A \cup B)' = A' \cap B'$ & $(A \cap B)' = A' \cup B'$
v) Law of contradiction: $A \cap A' = \Phi$
vi) Law of excluded middle: $A \cup A' = X$.

2.3.8 Properties of Fuzzy Sets

Using the definitions of union, intersection and complement of fuzzy sets we now prove the properties of fuzzy sets. It is seen that all the properties stated above for crisp sets holds good also for fuzzy sets except the law of contradiction and the law of excluded middle. In the following theorem we prove this.

Theorem 2.1 *Prove that for fuzzy sets commutative law, associative law, distributive law and De Morgan's law are true.*

Proof. We prove distributive law and De Morgan's law. Commutative law and associative law can be proved easily.

Let $\tilde{A}$ and $\tilde{B}$ and $\tilde{C}$ be fuzzy sets with membership functions $\mu_{\tilde{A}}(x)$, $\mu_{\tilde{B}}(x)\}$ and $\mu_{\tilde{C}}(x)$ respectively. We prove the distributive law $\quad \tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$
We have
$\mu_{\tilde{A} \cup (\tilde{B} \cap \tilde{C})}(x)$

$= \max\ [\mu_{\tilde{A}}(x), \mu_{\tilde{B}\cap\tilde{C}}(x)]$
$= \max\ [\mu_{\tilde{A}}(x), min\{\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)\}]$
$= \max\ [\alpha, min\{\beta, \gamma\}]$ where $\alpha = \mu_{\tilde{A}}(x), \beta = \mu_{\tilde{B}}(x)$ and $\gamma = \mu_{\tilde{C}}(x)$
$= f(x)$ [say]

Again

$\mu_{(\tilde{A}\cup\tilde{B})\cap(\tilde{A}\cup\tilde{C})}(x)$
$= \min\ [\mu_{\tilde{A}\cup\tilde{B}}(x),\ \mu_{\tilde{A}\cup\tilde{C}}(x)]$
$= \min\ [max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, max\{\mu_{\tilde{A}}(x), \mu_{\tilde{C}}(x)\}]$
$= \min\ [max\{\alpha, \beta\}, max\{\alpha, \gamma\}]$
$= g(x)$ [say]

For any fixed $x \in X$, there arise following six cases

i) $\alpha \le \beta \le \gamma$

ii) $\alpha \le \gamma \le \beta$

iii) $\beta \le \gamma \le \alpha$

iv) $\beta \le \alpha \le \gamma$

v) $\gamma \le \alpha \le \beta$

vi) $\gamma \le \beta \le \alpha$

We consider all these six cases in the following Table 2.2.

In all these six cases we see that $f(x) = g(x)$. This is true for any $x \in X$. Hence we have

Table 2.2: Different cases for distributive law of fuzzy sets.

Case	$\min\{\beta,\gamma\}$	$f(x)$	$\max\{\alpha,\beta\}$	$\max\ \{\alpha,\gamma\}$	$g(x)$
i) $\alpha \le \beta \le \gamma$	β	β	β	γ	β
ii) $\alpha \le \gamma \le \beta$	γ	γ	β	γ	γ
iii) $\beta \le \gamma \le \alpha$	β	α	α	α	α
iv) $\beta \le \alpha \le \gamma$	β	α	α	γ	α
v) $\gamma \le \alpha \le \beta$	γ	α	β	α	α
vi) $\gamma \le \beta \le \alpha$	γ	α	α	α	α

$\mu_{\tilde{A}\cup(\tilde{B}\cap\tilde{C})}(x) = \mu_{(\tilde{A}\cup\tilde{B})\cap(\tilde{A}\cup\tilde{C})}(x)$ for all $x \in X$.

This proves that

$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$

We now prove the De Morgan's law

$$(\tilde{A} \cup \tilde{B})' = \tilde{A}' \cap \tilde{B}'$$

We have

$\mu_{(\tilde{A}\cup\tilde{B})'}(x)$
$= 1 - \mu_{(\tilde{A}\cup\tilde{B})}(x)$
$= 1 - max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$
$= 1 - max\{\alpha, \beta\}$ where $\alpha = \mu_{\tilde{A}}(x)$ and $\beta = \mu_{\tilde{B}}(x)$

$= f(x)$ [say]

and $\mu_{\tilde{A}' \cap \tilde{B}'}(x)$

$= \min\{\mu_{\tilde{A}'}(x), \mu_{\tilde{B}'}(x)\} \qquad = \min\{1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x)\}$

$= \min\{1 - \alpha, 1 - \beta\}$

$= g(x)$ [say]

For any fixed $x \in X$, two cases will arise

Case (i) $\alpha \leq \beta$

Case (ii)$\alpha > \beta$

We consider these two cases in the following Table 2.3.

Table 2.3: Different cases for De Morgan's law of fuzzy sets.

Case	$max\{\alpha, \beta\}$	$f(x)$	$min\{1 - \alpha, 1 - \beta\}$	$g(x)$
i) $\alpha \leq \beta$	β	$1 - \beta$	$1 - \beta$	$1 - \beta$
ii)$\alpha > \beta$	α	$1 - \alpha$	$1 - \alpha$	$1 - \alpha$

In both the cases we see that $f(x) = g(x)$. This is true for any $x \in X$. Hence $(\tilde{A} \cup \tilde{B})' = \tilde{A}' \cap \tilde{B}'$

Theorem 2.2 *Prove that the law of contradiction and law of excluded middle do not hold for fuzzy sets.*

Proof. Law of contradiction is $\tilde{A} \cap \tilde{A}' = \Phi$

and law of excluded middle is $\tilde{A} \cup \tilde{A}' = X$

We have $\mu_{\tilde{A} \cap \tilde{A}'}(x)$

$= \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}'}(x)\}$

$= \min\{\mu_{\tilde{A}}(x), 1 - \mu_{\tilde{A}}(x)\}$

$= \min\{\alpha, 1 - \alpha\}$ where $\alpha = \mu_{\tilde{A}}(x)$

$\mu_{\phi}(x) = 0$ for $x \in X$

For any $\alpha \in (0, 1)$, $\min\{\alpha, 1 - \alpha\} \neq 0$

For $\alpha = 0$ and $\alpha = 1$ only $\min\{\alpha, 1 - \alpha\} = 0$

Thus $\min\{\alpha, 1 - \alpha\} = 0$ is not true in general for all $x \in X$.

i.e. $\mu_{\tilde{A} \cap \tilde{A}'}(x) = \mu_{\phi}(x)$ is not true in general for all $x \in X$.

i.e. $\tilde{A} \cap \tilde{A}' = \Phi$ is not true.

Again $\mu_{\tilde{A} \cup \tilde{A}'}(x)$

$= \max\{\mu_{\tilde{A}}(x), 1 - \mu_{\tilde{A}}(x)\}$

$= \max\{\alpha, 1 - \alpha\}$ where $\alpha = \mu_{\tilde{A}}(x)$

$$= \begin{cases} 1 & \text{for } \alpha = 0 \text{ } or \text{ } 1 \\ 1 - \alpha & \text{for } 0 < \alpha < 1/2 \\ \alpha & \text{for } 1/2 \leq \alpha < 1 \end{cases}$$

$\therefore \mu_{\tilde{A} \cup \tilde{A}'}(x) = 1$ is not true in general for all α i.e. for all $x \in X$.

i.e. $\tilde{A} \cup \tilde{A}' = X$ is not true.

Thus law of excluded middle is not true for fuzzy sets.

2.4 Some Definitions

To develop the notion of fuzzy numbers and for the study of the arithmetic of fuzzy numbers we need certain crisp sets associated with fuzzy sets under consideration. These crisp sets are called α- cut and is defined below .

2.4.1 Definition : $\alpha - cut$ of fuzzy set $\tilde{A}$.

The α- cut of the fuzzy set $\tilde{A}$ defined over the universal set X is the crisp set A_α defined by
$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$.

2.4.2 Support of a fuzzy set

Let $\tilde{A}$ be a fuzzy set in X. Then the support of $\tilde{A}$, denoted by $S(\tilde{A})$ is the crisp set defined by
$S(\tilde{A}) = \{x \in X : \mu_{\tilde{A}}(x) > 0\}$.

2.4.3 Height of a fuzzy set.

The height of a fuzzy set $\tilde{A}$ is defined as
$h(\tilde{A}) = sup\{\mu_{\tilde{A}}(x) : x \in X\}$

2.4.4 Normal fuzzy set

A fuzzy set $\tilde{A}$ is said to be normal if its height is one i.e. if $sup\{\mu_{\tilde{A}}(x) : x \in X\} = 1$.
If fuzzy set $\tilde{A}$ is not normal we can normalize it by redefining the membership function as $\mu_{\tilde{A}}(x)/h(\tilde{A})$, $x \in X$.

2.4.5 Convex fuzzy set

A fuzzy set $\tilde{A}$ in R^n is said to be a convex set if and only if for all $x_1, x_2 \in R^n$ and $0 \leq \lambda \leq 1$,
$\mu_{\tilde{A}}\{\lambda x_1 + (1-\lambda)x_2\} \geq min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$.

2.4.6 Fuzzy Number

A convex normal fuzzy set is called a fuzzy number.

The following theorem establishes a relation between the membership function and α-cuts of fuzzy set.

Theorem 2.3 *Let $\tilde{A}$ be a fuzzy set in X with the membership function $\mu_{\tilde{A}}(x)$. Let A_α be the α-cuts of $\tilde{A}$ and $\chi_{A_\alpha}(x)$ be the characteristic function of the crisp set A_α for $\alpha \in (0, 1]$. Then for each $x \in X$*

$$\mu_{\tilde{A}}(x) = sup\{\alpha \wedge \chi_{A_\alpha}(x) : 0 < \alpha \leq 1\}.$$

Proof. We have $\chi_{A_\alpha}(x) = \begin{cases} 1 \text{ if } x \in A_\alpha \\ 0 \text{ if } x \notin A_\alpha \end{cases}$

$\therefore$ For $x \in A_\alpha$ we have $\chi_{A_\alpha} = 1$ and $\mu_{\tilde{A}}(x) \geq \alpha$
and for $x \notin A_\alpha$ we have $\chi_{A_\alpha} = 0$ and $\mu_{\tilde{A}}(x) < \alpha$.
Now $sup\{\alpha \wedge \chi_{A_\alpha}(x) : 0 < \alpha \leq 1\}$

$= sup\ \{\alpha \wedge \chi_{A_\alpha}(x) : 0 < \alpha \leq \mu_{\tilde{A}}(x)\} \vee sup\ \{\alpha \wedge \chi_{A_\alpha}(x) : \mu_{\tilde{A}}(x) < \alpha \leq 1\}$
$= sup\ \{\alpha \wedge 1 : 0 < \alpha \leq \mu_{\tilde{A}}(x)\} \vee sup\ \{\alpha \wedge 0 : \mu_{\tilde{A}}(x) < \alpha \leq 1\}$
$= sup\ \{\alpha : 0 < \alpha \leq \mu_{\tilde{A}}(x)\}$
$= \mu_{\tilde{A}}(x)$

Remark : For given a fuzzy set $\tilde{A}$ in X we consider a special fuzzy set denoted by αA_α for $\alpha \in (0, 1]$ whose membership function is defined as

$\mu_{\alpha A_\alpha}(x) = \alpha \wedge \chi_{A_\alpha}(x)$ for all $x \in X$.

Let the set S_A be defined as

$S_A = \{\alpha : \mu_{\tilde{A}}(x) = \alpha\}$ for some $x \in X$.

We call this set as level set of $\tilde{A}$.

Result: From above theorem we now get the following theorem

Theorem 2.4 *The fuzzy set $\tilde{A}$ in X can be expressed in the form*

$$\tilde{A} = \bigcup\{\alpha A_\alpha : \alpha \in S_A\}$$

where $\bigcup$ denotes the standard fuzzy union.

This theorem is called the representation theorem of fuzzy sets. This theorem essentially tell that a fuzzy set $\tilde{A}$ in X can always be expressed in terms of its α-cuts without explicitly resorting to its membership function $\mu_{\tilde{A}}(x)$.

Theorem 2.4 is explained in the following example.

2.4.7 Example

Let $\tilde{A}$ be the fuzzy set defined by the membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & ,x \leq 1 \\ (x-1)/2 & ,1 < x \leq 3 \\ (5-x)/2 & ,3 < x \leq 5 \\ 0 & ,x > 5 \end{cases}$$

Its graph is shown in Fig. 2.1.

Let us consider four α-cuts viz. $A_{.2}, A_{.4}, A_{.6}$ and $A_{.8}$ in the following Fig. 2.1 and their corresponding membership functions $\mu_{.2A_{.2}}(x)$, $\mu_{.4A_{.4}}(x)$,$\mu_{.6A_{.6}}(x)$ are shown.

Finally, the union of these four fuzzy sets i.e. $(.2A_{.2}) \cup (.4A_{.4}) \cup (.6A_{.6}) \cup (.8A_{.8})$ is shown
i.e. $\bigcup\{\alpha A_\alpha : \alpha = .2, .4, .6, .8\}$ is shown.

It is seen that this is close to the graph of $\tilde{A}$ (Fig. 2.2)).

This explains the fact that if we consider all $\alpha \in (0, 1]$ then we get the graph of $\tilde{A}$ i.e.

$\bigcup\{\alpha A_\alpha : 0 < \alpha \leq 1\} = \tilde{A}$.

2.5 Some Useful and Important Fuzzy Numbers

Triangular Fuzzy Number: The graph of the membership function of triangular fuzzy number is of triangular shape. It is described by a triplet i.e. $\tilde{A} = (a_1, a_2, a_3)$, The membership function is given by

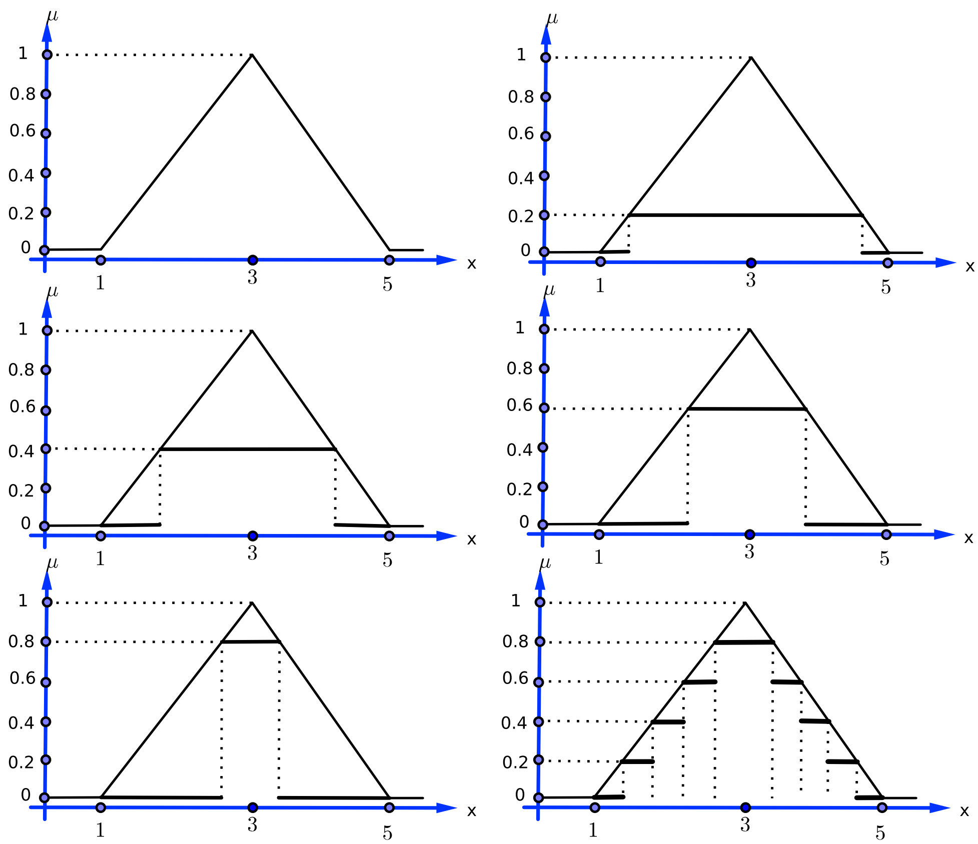

Figure 2.1: Example of representation theorem for fuzzy number

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x \leq a_1 \\ (x - a_1)/(a_2 - a_1) & \text{for } a_1 < x \leq a_2 \\ (a_3 - x)/(a_3 - a_2) & \text{for } a_2 \leq x < a_3 \\ 0 & \text{for } x \geq a_3 \end{cases}$$

The graph is shown in Fig. 2.2.

Trapezoidal Fuzzy Number: The graph of a trapezoidal fuzzy number is of the shape of a trapezium. It is described by a quadruplet i.e. $\tilde{A} = (a_1, a_2, a_3, a_4)$. The membership function of such a number is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x \leq a_1 \\ (x - a_1)/(a_2 - a_1) & \text{for } a_1 < x \leq a_2 \\ 1 & \text{for } a_2 < x \leq a_3 \\ (a_4 - x)/(a_4 - a_3) & \text{for } a_3 < x < a_4 \\ 0 & \text{for } x \geq a_4 \end{cases}$$

The graph of trapezoidal fuzzy number is shown in Fig. 2.3.

Rectangular Fuzzy Number (Interval Number) : Its graph looks like a rectangle. It is a special case of trapezoidal fuzzy number. It is nothing but an interval number. It is also represented by a

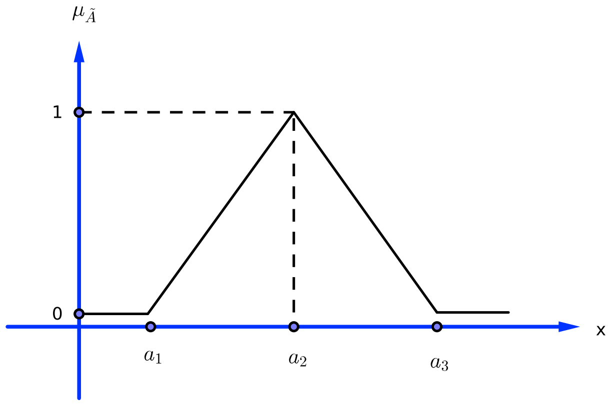

Figure 2.2: Membership function of triangular fuzzy number.

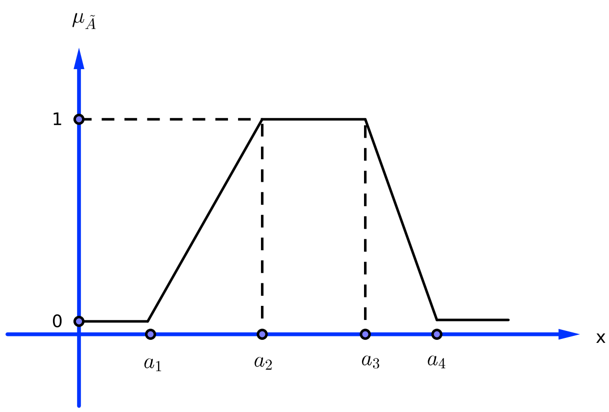

Figure 2.3: Membership function of trapezoidal fuzzy number.

quadruplet i.e. $\tilde{A} = (a_1, a_2, a_3, a_4)$ whose first two and last two components are alike i.e. $\tilde{A} = (a_1, a_1, a_2, a_2)$. The membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x < a_1 \\ 1 & \text{for } a_1 \leq x \leq a_2 \\ 0 & \text{for } x > a_2 \end{cases}$$

Its graph is shown in Fig. 2.4.

Note: We note the following important facts that

A trapezoidal number becomes $\tilde{A} = (a_1, a_2, a_3, a_4)$ becomes

i) a triangular number if $a_2 = a_3$.

ii) an interval or a rectangular number if $a_1 = a_2$ and $a_3 = a_4$

iii) a real number if $a_1 = a_2 = a_3 = a_4$.

Gaussian Fuzzy Number: It is described by a triplet $\tilde{A} = (m, \sigma_1, \sigma_2)$. The membership function is

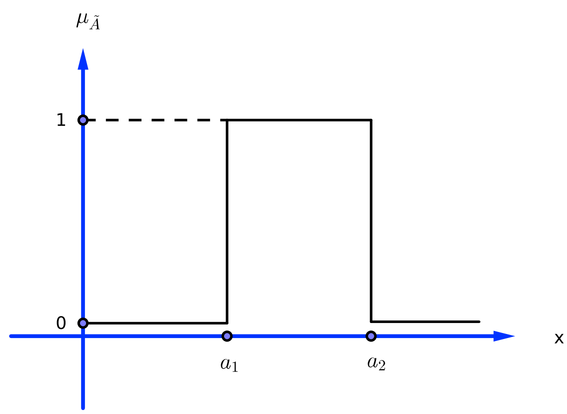

Figure 2.4: Membership function of rectangular fuzzy number.

given by

$$\mu_{\tilde{A}}(x) = \begin{cases} e^{-\frac{(x-m)^2}{2\sigma_1^2}} & \text{for } x \leq m \\ e^{-\frac{(x-m)^2}{2\sigma_2^2}} & \text{for } x > m \end{cases}$$

Its graph is shown in Fig. 2.5.

It is a symmetric curve about the line $x = m$ if $\sigma_1 = \sigma_2$. In general it is not symmetric.

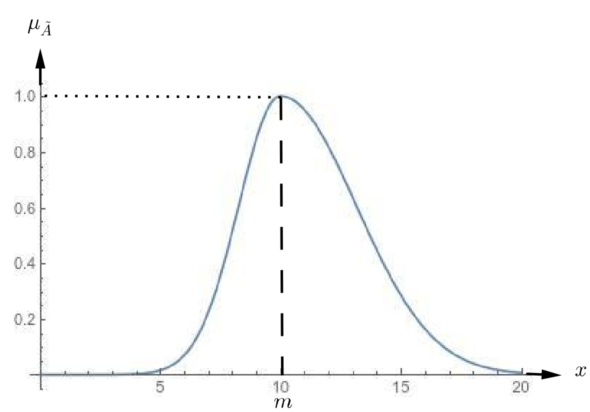

Figure 2.5: Membership function of rectangular fuzzy number.

2.6 Zadeh's Extension Principle

Zadeh's extension principle is a very important tool in fuzzy mathematics. This principle provides a procedure to fuzzify a crisp function. This type of fuzzification helps us to study mathematical relationships between fuzzy entities. Fuzzy arithmetic with fuzzy numbers is based on this principle.

Let $f : X \to Y$ be a crisp function. Let $P(X)$ and $P(Y)$ be the sets of all fuzzy sets of X and Y respectively. The function $f : X \to Y$ induces the function $f : P(X) \to P(Y)$ and the extension principle of Zadeh gives formulas to compute the membership function of the fuzzy set $f(\tilde{A})$ in Y in terms of the membership function of fuzzy set $\tilde{A}$ in X.

2.6.1 Definition: Zadeh's Extension Principle

Let $f : X \to Y$ be a mapping of the form $y = f(x)$ and $\tilde{A}$ be any fuzzy set of the fuzzy power set $P(X) of X$. If $\tilde{A}$ is mapped to $\tilde{B}$ by f i.e if $f(\tilde{A}) = \tilde{B}$ then the membership function of $\tilde{B}$ is given by

$$\mu_{f(\tilde{A})}(y) = \mu_{\tilde{B}}(y) = sup\{\mu_{\tilde{A}}(x) : x \in X, y = f(x)\}$$

More generally, let fuzzy sets $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ be defined on the universe $X_1, X_2, ..., X_n$ respectively. The mapping $f : X_1 \times X_2 \times ... \times X_n \to Y$ of the form $f(x_1, x_2, ..., x_n) = y$ allows us to determine the membership function of the fuzzy set $f(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \tilde{B}$ as follows by the extension principle.

$$\mu_{\tilde{B}}(y) = sup\ [\ min\{\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2), ..., \mu_{\tilde{A}_n}(x_n)\} : y = f(x_1, x_2, ..., x_n)\ ]$$

Zadeh's extension principle is a very powerful idea and is one of the fundamentals of fuzzy set theory. It gives us the rule of calculation of an output of a fuzzy system when we know the structure of the input fuzzy system.

2.7 Arithmetic of Fuzzy Numbers

Using Zadeh's extension principle, the arithmetic operations on fuzzy numbers are defined as follows.

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers then addition, subtraction, multiplication and division are defined as follows.

Addition: $\mu_{\tilde{A}+\tilde{B}}(z) = \sup_{z=x+y} [\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\ ]$

Subtraction: $\mu_{\tilde{A}-\tilde{B}}(z) = \sup_{z=x-y} [\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\ ]$

Multiplication: $\mu_{\tilde{A}.\tilde{B}}(z) = \sup_{z=x.y} [\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\ ]$

Division: $\mu_{\tilde{A}/\tilde{B}}(z) = \sup_{z=x/y} [\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\ ]$

77.7.1 Example. Using the addition rule for fuzzy numbers show that 3+7=10 for real numbers.

Solution. We know that every real number is a particular fuzzy number. Let $3 = \tilde{A}$ & $7 = \tilde{B}$.

Then their membership functions are

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{for } x = 3 \\ 0 & \text{for } x \neq 3 \end{cases}$$

and

$$\mu_{\tilde{B}}(y) = \begin{cases} 1 & \text{for } y = 7 \\ 0 & \text{for } y \neq 7 \end{cases}$$

From definition the membership function of $\tilde{A} + \tilde{B}$ is given by

$$\mu_{\tilde{A}+\tilde{B}}(z) = \sup_{z=x+y} [\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\ ]$$

For z = 10 we have $\mu_{\tilde{A}+\tilde{B}}(z) = \sup_{x+y=10} [\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\ ]$

x	$\mu_{\tilde{A}}(x)$	$\mu_{\tilde{B}}(10-x)$	$\beta(x)$
$x=3$	1	1	1
$x\neq 3$	0	0	0

$$= \sup_x[\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(10-x)\}\ ] = \sup_x[\ \beta(x)\ ] \text{ (say)}$$

$\therefore \mu_{\tilde{A}+\tilde{B}}(z) = sup\ \{1, 0\} = 1$

For $z \neq 10$ we have $\mu_{\tilde{A}+\tilde{B}}(z) = \sup_x[\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(z-x)\}\ ] = \sup_x[\ \beta(x)\ ]$ (say)

x	$\mu_{\tilde{A}}(x)$	$\mu_{\tilde{B}}(z-x)$	$\beta(x)$
$x=3$	1	0	0
$x\neq 3$	0	0 or 1	0

$\therefore \mu_{\tilde{A}+\tilde{B}}(z) = sup\ \{0, 0\} = 0$

Hence we have

$$\mu_{\tilde{A}+\tilde{B}}(z) = \begin{cases} 1\ for\ z = 10 \\ 0\ for\ z \neq 10 \end{cases}$$

This proves that $\tilde{A}+\tilde{B} = 10$

77.7.2 Example. Using addition rule for fuzzy numbers, prove that $[3,5]+[4,8] = [7,13]$

Solution. Let $\tilde{A} = [3,5]$, $\tilde{B} = [4,8]$ and $\tilde{C} = [7,13]$

Then

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x < 3 \\ 1 & \text{for } 3 \leq x \leq 5 \\ 0 & \text{for } x > 5 \end{cases} ; \quad \mu_{\tilde{B}}(y) = \begin{cases} 0 & \text{for } y < 4 \\ 1 & \text{for } 4 \leq y \leq 8 \\ 0 & \text{for } y > 8 \end{cases} \text{ and } \mu_{\tilde{C}}(z) = \begin{cases} 0 & \text{for } z < 7 \\ 1 & \text{for } 7 \leq z \leq 13 \\ 0 & \text{for } z > 13 \end{cases}$$

We have to prove that $\tilde{A}+\tilde{B} = \tilde{C}$

From the addition rule for fuzzy numbers we have

$$\begin{aligned} \mu_{\tilde{A}+\tilde{B}}(z) &= \sup_{x+y=z}[\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\ ] \\ &= \sup_{x+y=z}[\ min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(z-x)\}\ ] \\ &= \sup_{x+y=z}[\ g(x)\ ], \text{ where } g(x) = min\ \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(z-x)\}. \end{aligned}$$

For any $z < 7$ we have

x	$\mu_{\tilde{A}}(x)$	$\mu_{\tilde{B}}(z-x)$	$g(x)$
$x<3$	0	0 or 1	0
$3 \leq x \leq 5$	1	0 as $-\infty < z-x < 4$	0
$x>5$	0	0 as $-\infty < z-x < 2$	0

$\therefore$ For any $z < 7$ we have $\mu_{\tilde{A}+\tilde{B}}(z) = sup\{0,0,0\} = 0$

For any z with $7 \leq z \leq 13$ we have

$\therefore$ When $7 \leq z \leq 13$ then $\mu_{\tilde{A}+\tilde{B}}(z) = sup\{0,1\} = 1$

For any z with $z > 13$ we have

x	$\mu_{\tilde{A}}(x)$	$\mu_{\tilde{B}}(z-x)$	$g(x)$
$x < 3$	0	0 or 1 as $4 \leq z - x < \infty$	0
$3 \leq x \leq 5$	1	0 or 1 as $2 < z - x < 10$	0 or 1
$x > 5$	0	0 or 1 as $-\infty < z - x < \infty$	0

x	$\mu_{\tilde{A}}(x)$	$\mu_{\tilde{B}}(z-x)$	$g(x)$
$x < 3$	0	0 as $10 < z - x < \infty$	0
$3 \leq x \leq 5$	1	0 as $8 < z - x < \infty$	0
$x > 5$	0	0 or 1 as $-\infty < z - x < \infty$	0

$\therefore$ When $z > 13$ then $\mu_{\tilde{A}+\tilde{B}}(z) = sup\{0, 0, 0\} = 0$
Thus

$$\mu_{\tilde{A}+\tilde{B}}(z) = \begin{cases} 0 & \text{for } z < 7 \\ 1 & \text{for } 7 \leq z \leq 13 \\ 0 & \text{for } z > 13 \end{cases}$$

Note: In general using addition, subtraction, multiplication and division rule for fuzzy numbers we can prove the laws of addition, subtraction, multiplication and division for intervals. This is shown below.

2.8 Arithmetic Operations on Fuzzy Numbers using α-cuts.

In Theorem 2.4, we have proved that any fuzzy number can be expressed in the form
$\tilde{A} = \bigcup\{\alpha A_\alpha : \alpha \in S_A\}$
where αA_α is a special fuzzy set with membership function
$\mu_{\alpha A_\alpha}(x) = \alpha \wedge \chi_{A_\alpha}(x)$ for all $x \in X$.
Since fuzzy number is normal convex set, it follows that αA_α is nothing but an interval number.
We denote it by $(\tilde{A})_\alpha$.
Thus above result becomes $\tilde{A} = \bigcup\{(\tilde{A})_\alpha : 0 < \alpha \leq 1\}$
To perform arithmetic operations on fuzzy numbers using this result we proceed as follows.
Let $\tilde{A}$ and $\tilde{B}$ denote fuzzy numbers and * denote any of the four basic arithmetic operations $\{+, -, ., \div\}$.
Thus for $\tilde{A} = \bigcup\{(\tilde{A})_\alpha : 0 < \alpha \leq 1\}$ and $\tilde{B} = \bigcup\{(\tilde{B})_\alpha : 0 < \alpha \leq 1\}$.
Using above result we get $\tilde{A} * \tilde{B} = \bigcup\{(\tilde{A})_\alpha * \tilde{B})_\alpha : 0 < \alpha \leq 1\}$.

77.8.1 Example : Let us consider two triangular numbers $\tilde{A}$ and $\tilde{B}$ with membership functions given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ (x+1)/2 & \text{for } -1 < x \leq 1 \\ (3-x)/2 & \text{for } 1 < x \leq 3 \\ 0 & \text{for } x > 3 \end{cases}$$

$$\mu_{\tilde{B}}(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ (x-1)/2 & \text{for } 1 < x \leq 3 \\ (5-x)/2 & \text{for } 3 < x \leq 5 \\ 0 & \text{for } x > 5 \end{cases}$$

To get α-cut of $\tilde{A}$ we have from $\mu_{\tilde{A}}(x)$
$(x+1)/2 = \alpha$ & $(3-x)/2 = \alpha$
i.e. $x = 2\alpha - 1$ & $x = 3 - 2\alpha$
$\therefore$ α-cut of $\tilde{A}$ is $(\tilde{A})_\alpha = [2\alpha - 1, 3 - 2\alpha]$.
Similarly, to get α-cut of $\tilde{B}$ we have from $\mu_{\tilde{B}}(x)$
$(x-1)/2 = \alpha$ & $(5-x)/2 = \alpha$
i.e. $x = 2\alpha + 1$ & $x = 5 - 2\alpha$
$\therefore$ α-cut of $\tilde{B}$ is $(\tilde{B})_\alpha = [2\alpha + 1, 5 - 2\alpha]$.
Now $(\tilde{A})_\alpha + (\tilde{B})_\alpha = [2\alpha - 1, 3 - 2\alpha] + [2\alpha + 1, 5 - 2\alpha]$
$= [4\alpha, 8 - 4\alpha]$
$(\tilde{A})_\alpha - (\tilde{B})_\alpha = [2\alpha - 1, 3 - 2\alpha] - [2\alpha + 1, 5 - 2\alpha]$
$= [4\alpha - 6, 2 - 4\alpha]$
$\therefore$ $\tilde{A} + \tilde{B} = \bigcup\{[4\alpha, 8 - 4\alpha] : 0 < \alpha \leq 1\}$
and $\tilde{A} - \tilde{B} = \bigcup\{[4\alpha - 6, 2 - 4\alpha] : 0 < \alpha \leq 1\}$
Now $4\alpha = x$ gives $\alpha = x/4$
and $8 - 4\alpha = x$ gives $\alpha = (8 - x)/4$
$\therefore$

$$\tilde{A} + \tilde{B} = \begin{cases} 0 & \text{for } x \leq 1 \\ x/4 & \text{for } 0 < x \leq 4 \\ (8-x)/4 & \text{for } 4 < x < 8 \\ 0 & \text{for } x \geq 8 \end{cases}$$

Again $4\alpha - 6 = x$ gives $\alpha = (x + 6)/4$
$2 - 4\alpha = x$ gives $\alpha = (2 - x)/4$
$\therefore$

$$\tilde{A} - \tilde{B} = \begin{cases} 0 & \text{for } x \leq 6 \\ (x+6)/4 & \text{for } -6 < x \leq -2 \\ (2-x)/4 & \text{for } -2 < x < 2 \\ 0 & \text{for } x \geq 2 \end{cases}$$

In terms-of triplet notation we see that
$\tilde{A} = [-1, 1, 3]$, $\tilde{B} = [1, 3, 5]$
and $\tilde{A} + \tilde{B} = [0, 4, 8]$, $\tilde{A} - \tilde{B} = [-6, -2, 2]$
Note: We see that $\tilde{A} + \tilde{B} = [-1 + 1, 1 + 3, 3 + 5]$
and$\tilde{A} - \tilde{B} = [-1 - 5, 1 - 3, 3 - 1]$

77.8.1 Rules for addition subtraction and scalar multiplication of triangular fuzzy numbers

Theorem: If $\tilde{A} = [a_1, b_1, c_1]$ and $\tilde{B} = [a_2, b_2, c_2]$ then prove that
$\tilde{A} + \tilde{B} = [a_1 + a_2, b_1 + b_2, c_1 + c_2]$, $\tilde{A} - \tilde{B} = [a_1 - c_2, b_1 - b_2, c_1 - a_2]$ and

$$k\tilde{A} = \begin{cases} [ka_1, kb_1, kc_1] & fork \geq 0 \\ [kc_1, kb_1, ka_1] & fork < 0 \end{cases}$$

Proof. Here $\tilde{A} = [a_1, b_1, c_1]$ and $\tilde{B} = [a_2, b_2, c_2]$.
The membership functions of $\tilde{A}$ and $\tilde{B}$ are given respectively by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x \leq a_1 \\ (x - a_1)/(b_1 - a_1) & \text{for } a_1 < x \leq b_1 \\ (c_1 - x)/(c_1 - b_1) & \text{for } b_1 < x < c_1 \\ 0 & \text{for } x \geq c_1 \end{cases}$$

$$\mu_{\tilde{B}}(x) = \begin{cases} 0 & \text{for } x \le a_2 \\ (x-a_2)/(b_2-a_2) & \text{for } a_2 < x \le b_2 \\ (c_2-x)/(c_2-b_2) & \text{for } b_2 < x < c_2 \\ 0 & \text{for } x \ge c_2 \end{cases}$$

To get α-cut of $\tilde{A}$ we have from $\mu_{\tilde{A}}(x)$
$(x-a_1)/(b_1-a_1) = \alpha$ and $(c_1-x)/(c_1-b_1) = \alpha$
From these $x = \alpha(b_1-a_1)+a_1$ and $x = c_1 - \alpha(c_1-b_1)$.
$\therefore (\tilde{A})_\alpha = [\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)]$
To get α-cut of $\tilde{B}$ we have from $\mu_{\tilde{B}}(x)$
$(x-a_2)/(b_2-a_2) = \alpha$ and $(c_2-x)/(c_2-b_2) = \alpha$
From these $x = \alpha(b_2-a_2)+a_2$ and $x = c_2 - \alpha(c_2-b_2)$.
$\therefore (\tilde{B})_\alpha = [\alpha(b_2-a_2)+a_2, c_1-\alpha(c_2-b_2)]$
Using addition role for Interval numbers we get
$(\tilde{A})_\alpha + (\tilde{B})_\alpha = [\alpha(b_1+b_2-a_1-a_2)+a_1+a_2, c_1+c_2-\alpha(c_1+c_2-b_1-b_2)]$
$\therefore \tilde{A}+\tilde{B} = \bigcup\{[\alpha(b_1+b_2-a_1-a_2)+a_1+a_2, c_1+c_2-\alpha(c_1+c_2-b_1-b_2)] : 0 < \alpha \le 1\}$
From $\alpha(b_1+b_2-a_1-a_2)+a_1+a_2 = x$ we have
$\alpha = (x-a_1-a_2)/(b_1+b_2-a_1-a_2)$
From $x = c_1+c_2-\alpha(c_1+c_2-b_1-b_2)$ we have
$\alpha = (c_1+c_2-x)/(c_1+c_2-b_1-b_2)$
Hence we have

$$\mu_{\tilde{A}+\tilde{B}}(x) = \begin{cases} 0 & \text{for } x \le a_1+a_2 \\ (x-a_1-a_2)/(b_1+b_2-a_1-a_2) & \text{for } a_1+a_2 < x \le b_1+b_2 \\ (c_1+c_2-x)/(c_1+c_2-b_1-b_2) & \text{for } b_1+b_2 < x < c_1+c_2 \\ 0 & \text{for } x \ge c_1+c_2 \end{cases}$$

i.e. $\tilde{A}+\tilde{B} = [a_1+a_2, b_1+b_2, c_1+c_2]$
To get subtraction rule we have

$$\begin{aligned}(\tilde{A})_\alpha - (\tilde{B})_\alpha &= [\alpha(b_1-a_1)+a_1) - c_2 + \alpha(c_2-b_2), c_1-\alpha(c_1-b_1)-\alpha(b_2-a_2)-a_2] \\ &= [\alpha(b_1-a_1+c_2-b_2)+a_1) - c_2, c_1-a_2-\alpha(c_1-b_1+b_2-a_2)]\end{aligned}$$

$\therefore \tilde{A}-\tilde{B} = \bigcup\{[\alpha(b_1-a_1+c_2-b_2)+a_1)-c_2, c_1-a_2-\alpha(c_1-b_1+b_2-a_2)] : 0 < \alpha \le 1\}$
From $\alpha(b_1-a_1+c-b_2)+a_1)-c_2 = x$ we get
$\alpha = (x-a_1+c_2)/(b_1+c_2-a_1-b_2)$
From $c_1-a_2-\alpha(c_1-b_1+b_2-a_2 = x$ we get
$\alpha = (x-c_1+a_2)/(c_1-b_1+b_2-a_2)$

$$\mu_{\tilde{A}-\tilde{B}}(x) = \begin{cases} 0 & \text{for } x \le a_1-c_2 \\ (x-a_1+c_2)/(b_1+c_2-a_1-b_2) & \text{for } a_1-c_2 < x \le b_1-b_2 \\ (x-c_1+a_2)/(c_1-b_1+b_2-a_2) & \text{for } b_1-b_2 < x < c_1-a_2 \\ 0 & \text{for } x \ge c_1-a_2 \end{cases}$$

i.e. $\tilde{A}-\tilde{B} = [a_1-c_2, b_1-b_2, c_1-a_2]$
To prove the scalar multiplication rule we note that the α-cut of $\tilde{A}$ is
$(\tilde{A})_\alpha = [\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)]$
From the rule of scalar multiplication of intervals we have

$$k(\tilde{A})_\alpha = \begin{cases} [k\alpha(b_1 - a_1) + ka_1, kc_1 - k\alpha(c_1 - b_1)] & fork \geq 0 \\ [kc_1 - k\alpha(c_1 - b_1), k\alpha(b_1 - a_1) + ka_1] & fork < 0 \end{cases}$$

$\therefore$ For $k \geq 0$ we have
$k\tilde{A} = \bigcup\{k\alpha(b_1 - a_1) + ka_1, kc_1 - k\alpha(c_1 - b_1)] : 0 < \alpha \leq 1\}$
Now $x = k\alpha(b_1 - a_1) + ka_1$ and $x = kc_1 - k\alpha(c_1 - b_1)$ gives
$\alpha = (x - ka_1)/k(b_1 - a_1)$ and $\alpha = (kc_1 - x)/k(c_1 - b_1)$.
$\therefore$ For $k \geq 0$ we have

$$\mu_{k\tilde{A}}(x) = \begin{cases} 0 & \text{for } x \leq ka_1 \\ (x - ka_1)/(k(b_1 - a_1) & \text{for } ka_1 < x < kb_1 \\ (kc_1 - x)/(k(c_1 - b_1) & \text{for } kb_1 < x < kc_1 \\ 0 & \text{for } x \geq kc_1 \end{cases}$$

This gives $k\tilde{A} = [ka_1, kb_1, kc_1]$.
Similarly for $k < 0$ we have
$k\tilde{A} = \bigcup\{[kc_1 - k\alpha(c_1 - b_1), k\alpha(b_1 - a_1) + ka_1]\}$
Now $x = kc_1 - k\alpha(c_1 - b_1)$ and $x = k\alpha(b_1 - a_1) + ka_1$ gives
$\alpha = (kc_1 - x)/k(c_1 - b_1)$ and $\alpha = (x - ka_1))/k(b_1 - a_1)$.
$\therefore$ For $k < 0$ we have

$$\mu_{k\tilde{A}}(x) = \begin{cases} 0 & \text{for } x \leq kc_1 \\ (kc_1 - x)/k(c_1 - b_1) & \text{for } kc_1 < x < kb_1 \\ (x - ka_1))/k(b_1 - a_1) & \text{for } kb_1 < x < ka_1 \\ 0 & \text{for } x \geq ka_1 \end{cases}$$

This gives $k\tilde{A} = [kc_1, kb_1, ka_1]$.
Hence we get finally $k\tilde{A} = \begin{cases} [ka_1, kb_1, kc_1] & fork \geq 0 \\ [kc_1, kb_1, ka_1] & fork < 0 \end{cases}$

77.8.3 Rules for addition subtraction and scalar multiplication of trapezoidal fuzzy number

Proceeding exactly in the same way as triangular fuzzy numbers we can easily prove the following rules.
Addition:If $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ then
$\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$
Subtraction: If $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ then
$\tilde{A} - \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$
Scalar Multiplication: If $\tilde{A} = (a_1, a_2, a_3, a_4)$ and k is a scalar then

$$k\tilde{A} = \begin{cases} [ka_1, ka_2, ka_3, ka_4] & fork \geq 0 \\ [ka_4, ka_3, ka_2, ka_1] & fork < 0 \end{cases}$$

77.8.4 Rules for addition and subtraction for Gaussian fuzzy numbers

Addition: If $\tilde{A} = (m_a, \sigma_{1a}, \sigma_{2a})$ and $\tilde{B} = (m_b, \sigma_{1b}, \sigma_{2b})$ then
$\tilde{A} + \tilde{B} = (m_a + m_b, \sigma_{1a} + \sigma_{1b}, \sigma_{2a} + \sigma_{2b})$
Subtraction: If $\tilde{A} = (m_a, \sigma_{1a}, \sigma_{2a})$ and $\tilde{B} = (m_b, \sigma_{1b}, \sigma_{2b})$ then
$\tilde{A} - \tilde{B} = (m_a - m_b, \sigma_{1a} - \sigma_{1b}, \sigma_{2a} - \sigma_{2b})$

77.9 Illustrative Examples

77.9.1 Example. Show that for interval numbers distributive law does not hold in general.
Solution. Let $X = [1, 4], Y = [2, 5]$ and $Z = [3, 8]$.

$\therefore X + Y = [1 + 2, 4 + 5] = [3, 9]$
Now $(X + Y)Z$
$= [3, 9]\ [3, 8]$
$= [9, 72]$
and $XZ + YZ$
$= [3, 32] + [6, 40]$
$= [9, 72]$
$\therefore (X + Y)Z = XZ + YZ$
Again let $X = [-1, 3]$, $Y = (2, 4]$, $Z = [-3, -1]$
$\therefore (X + Y)Z$
$= [1, 7]\ [-3, -1]$
$= [min\{-3, -1, -21, -7\}, max\{-3, -1, -21, -7\}]$
$= [-21, -1]$
$XZ + YZ$
$= [min\{3, 1, -9, -3\}, max\{3, 1, -9, -3\}] + [min\{-6, -2, -12, -4\}, max\{-6, -2, -12, -4\}]$
$= [-9, 3] + [-12, -2]$
$= [-21, 1]$
$\therefore (X + Y)Z \subset XZ + YZ$
Hence distributive law does not hold in general.

77.9.2 Example. Show that the fuzzy set with following membership function is neither normal nor convex.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 \quad for\ x \leq 1 \\ 3(x-1)/8 \quad for\ 1 < x \leq 3 \\ (6-x)/4 \quad for\ 3 < x \leq 4 \\ (3x-2)/20 \quad for\ 4 < x \leq 6 \\ 3(7-x)/5 \quad for\ 6 < x < 7 \\ 0 \quad for\ x \geq 7 \end{cases}$$

Solution. We first draw the graph of this fuzzy set.

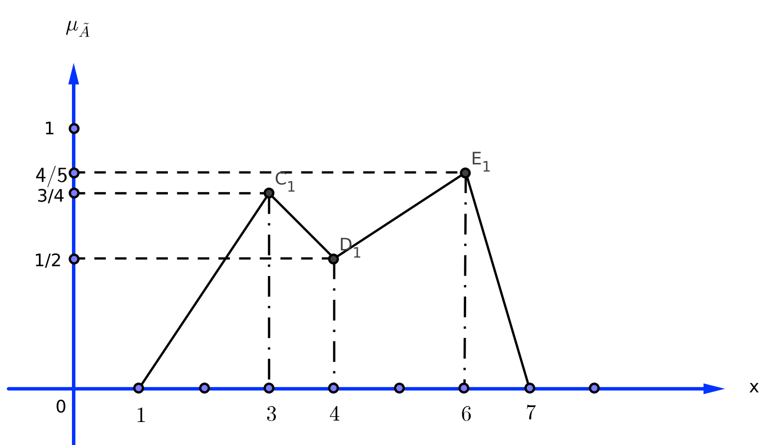

Figure 2.6: Non convex and normal fuzzy set.

From the figure we see that the height of this fuzzy set is 4/5 which is less than one. Hence the fuzzy set is not normal.

To show that the fuzzy set is non-convex we consider two points $x_1 = 3$ and $x_2 = 6$.

Now $\mu_{\tilde{A}}(x_1) = 3/4$ and $\mu_{\tilde{A}}(x_2) = 4/5$

$\therefore min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} = \{3/4, 4/5\} = 3/4$

Again for $\lambda = 2/3$

$\lambda x_1 + (1-\lambda)x_2 = \frac{2}{3} \times 3 + \frac{1}{3} \times 6 = 4$

$\therefore \mu_{\tilde{A}}\{\lambda x_1 + (1-\lambda)x_2\} = \mu_{\tilde{A}}(4) = 1/2$

But $\frac{1}{2} < \frac{3}{4} \therefore \mu_{\tilde{A}}\{\lambda x_1 + (1-\lambda)x_2\} < min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$

This shows that $\tilde{A}$ is not convex set.

77.9.3 Example. Evaluate the following

$2(5, 6, 8, 12) + 3(-1, 3, 4) - 5[-3, 2] + 8.$

Solution. $2(5, 6, 8, 12) + 3(-1, 3, 4) - 5[-3, 2] + 8.$

$=2(5, 6, 8, 12) + 3(-1, 3, 3, 4) - 5(-3, -3, 2, 2) + 8(1, 1, 1, 1)$

$=(10, 12, 16, 24) + (-3, 9, 9, 12) - (-15, -15, 10, 10) + (8, 8, 8, 8)$

$=(15, 29, 33, 44) - (-15, -15, 10, 10)$

$=(15 - 10, 29 - 10, 33 + 15, 44 + 15)$

$=(5, 19, 48, 59).$

77.10 Summary In this module the notion of interval numbers is introduced first. Then operations on fuzzy sets are introduced. Fuzzy numbers are defined and arithmetic operations on them are discussed. The famous extension principle of Zadeh is taken into account for this purpose. The arithmetic operations on fuzzy numbers are considered as an extension of arithmetic operations on interval numbers by representing fuzzy numbers as union of α-cuts. All these are illustrated with the help of examples.

2.9 References

77.11 Suggested Further Readings

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