Lecture 1: Proton-proton scattering at low energy
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Proton-proton (p-p) scattering at low energy:

The stability of nuclei containing protons and neutrons shows that a strong and short-range attractive force must be present between protons at close distance, just like the neutron-proton force. In a nucleus of proton number $z > 1$, the protons create a long-range Coulomb repulsive force due to their $+ve$ charges. So, the strong and short-range attractive force and the long-range repulsive force are always present inside this nucleus. But, the short-range attractive force is actually dominates over the long-range Coulombian repulsive force within the nucleus, this results the stability of the nuclei.

Since, no stable bound state of two protons is observed, the only means of the investigating the p-p force is through the experiments on p-p scattering.

The investigations on the theory of p-p scattering of give the information about the strength and range of the p-p forces. The analysis on the p-p scattering shows that the nucleon potential is charge independent. That is same nuclear potential may be used for both p-p and n-p scattering, but their cross-sections are not the same because cross-section depends on energy.

The p-p scattering is caused not only by the nuclear forces but also by the Coulomb force. For incident protons of energy below $10\ MeV$, only the S-state interaction is of very importance in the scattering, since protons in higher orbital angular momentum states stay apart from each other beyond the range of the nuclear force (b).

Experimental study on the p-p scattering is capable of much higher accuracy than the n-p scattering, and the p-p scattering experiments are easier to perform and interpret due to the following reasons:

(i) Protons are easily available over a wide range of energies.
(ii) Protons can be produced in well collimated beam.
(iii) Protons can be made monoenergetic by different kinds of accelerators.
(iv) Protons can easily be detected by their ionizing properties, and their energies can be measured more easily.
(v) Protons undergo both Coulomb and nuclear scattering. This increases the sensitivity in case one of the scattering probabilities is small, and gives the sign of the phase-shift resulting from the nuclear scattering.
(vi) The protons combination obeys the Fermi-Dirac statistics. This simplifies the analysis of p-p scattering.

From the **theoretical stand point**, the p-p scattering calculations are more complicated for the following reasons:

(i) The presence of Coulomb scattering in addition to nuclear scattering introduces interference effect. This interference phenomena permits to determine the sign of the phase-shift for p-p scattering. The Coulomb potential appreciably distorts the incident wave even at finite distances. The Coulomb scattering calculations require special wave mechanical treatment because of the slow variation of the potential with distance. So, the theory is more complicated than the of n-p scattering.
(ii) Effect of indistinguishability of the scattering and scattered particles. Because, the scattering and scattered particles are identical and they obey the Pauli's exclusion

principle. The nucleons have spin $\frac{1}{2}$, and hence their wave functions must be antisymmetric with respect to interchange of the nucleons. Therefore, the wave describing the two protons must change sign on the interchange of the two particles: scattering and scattered particles.

We are interest to obtain a theoretical expression for the differential elastic scattering cross-section of protons by protons. Partial wave method for calculating the cross-section is applicable only when the potential of the interaction is of the form:

$$V(r) \sim \frac{1}{r^n} \qquad with \qquad n > 1$$

Since, the Coulomb potential $V(r) \propto 1/r$ is not possible to apply in the method of partial waves in this case and hence the Coulomb scattering calculations are made using parabolic coordinates.

For the particles incident along the Z-axis, the asymptotic wave function in the presence of Coulomb interaction is

$$\psi(r,\theta) = \exp\{ikz + i\eta \ln k(r-z)\} + \frac{f_c(\theta)}{r}\exp\{i(kr - \eta \ln 2kr + 2\eta_0 + \pi)\} \qquad \dots(1)$$

Here, θ is the scattering angle (Fig. 1),

$$\eta = \frac{e^2}{4\pi\epsilon_0 \hbar v}$$

$$k = \frac{\sqrt{ME}}{\hbar} = \frac{Mv}{2\hbar}$$

$$f_c(\theta) = \frac{\eta}{2k\, sin^2\left(\frac{\theta}{2}\right)}\exp\left\{-i\eta \ln sin^2\left(\frac{\theta}{2}\right)\right\}$$

$$\eta_0 = \arg\Gamma(1 + i\eta)$$

First term in equation (1) represents the **incident plane wave** and second term represents **spherical outgoing wave**. These two waves are **slightly distorted** due to the **effect of Coulomb potential**. The term $f_c(\theta)$ is the Coulomb scattering amplitude. It is related with the differential scattering cross-section $\sigma_c(\theta)$ for Coulomb scattering in C-system as given by

$$\sigma_c(\theta) = |f_c(\theta)|^2$$

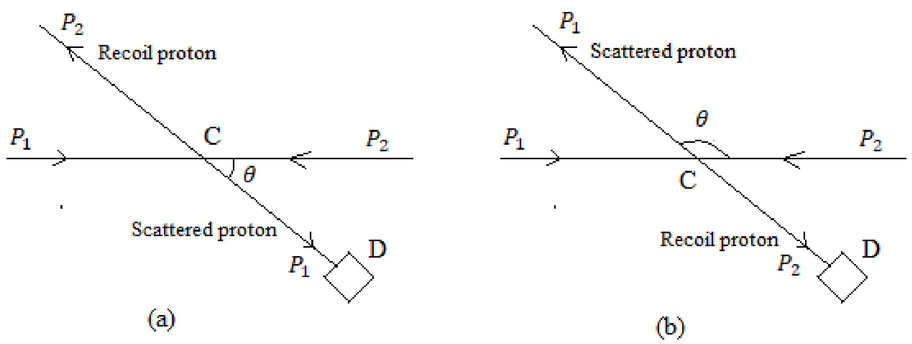

Fig. 1: Detection of scattered and recoil protons by the same detector D in p-p scattering

The expression for the scattering cross-section $\sigma(\theta)$ when the proton scattered at angle θ is given by

$$\sigma(\theta) = |f_c(\theta)|^2$$

$$= \left[\frac{\eta}{2k\, sin^2\left(\frac{\theta}{2}\right)} \exp\left\{ -i\eta \ln sin^2\left(\frac{\theta}{2}\right) \right\} \right]^2$$

This becomes

$$\sigma(\theta) = \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \frac{1}{sin^4(\theta/2)} \qquad \dots (2)$$

This is the Rutherford scattering formula applied in the case of p-p scattering. Rutherford obtained it from **classical considerations**.

If we neglecting the spin of the two protons, then the two protons are identical and hence it is not possible for the detector D to distinguish between the incident proton which is scattered at θ and the recoil proton when the incident proton scattered at $(\pi - \theta)$, this is shown in Fig. 1.

For the proton scattered at $(\pi - \theta)$, we can write

$$\sigma(\pi - \theta) = \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \frac{1}{sin^4\left(\frac{\pi - \theta}{2}\right)}$$

$$= \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \frac{1}{cos^4\left(\frac{\theta}{2}\right)} \qquad \dots (3)$$

So, the total scattering cross-section in the C-system is

$$\sigma_c(\theta) = \sigma(\theta) + \sigma(\pi - \theta)$$

$$= \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \left[\frac{1}{sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{cos^4\left(\frac{\theta}{2}\right)}\right] \qquad \dots (4)$$

The equation (4) is obtained from **classical considerations**.

Now from the kinematics of the collision process, we have seen that the angle of scattering (θ_L) in the L-system is related to that in the C-system as θ_C for p-p scattering by the equation

$$\theta_C = 2\theta_L$$

Also, if the scattering cross-section in the L-system is defined by $\sigma_L(\theta_L)$, then

$$\sigma_c(\theta) = \frac{\sigma_L(\theta_L)}{4 \cos\theta_L}$$

$$or, \quad \sigma_L(\theta_L) = 4\cos\theta_L . \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \left[\frac{1}{sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{cos^4\left(\frac{\theta}{2}\right)}\right] \quad \dots (5)$$

For wave mechanical treatment of the problem, we have to take into account the exchange effect which is associated with the indistinguishability of the two protons, and add up the amplitudes of the scattered waves at θ and $(\pi - \theta)$ and not the modulus squared of the amplitudes as was done in the classical considerations. Here, exchange of the two proton coordinates means the exchanging θ with $(\pi - \theta)$.

For identical particles, their waves interfere and instead of the summing square of the amplitudes, the amplitudes are sum up first and then we square them. Thus the eigen functions containing only spatial coordinates without spin will be

$$f(\theta) \pm f\big((\pi - \theta)\big)$$

The space part of the wave function $f(\theta) + f\big((\pi - \theta)\big)$ is symmetric and $f(\theta) - f\big((\pi - \theta)\big)$ is antisymmetric with respect to the exchange of coordinates.

Now, for protons, the total eigen function must be antisymmetric, hence the resultant scattering amplitude will be $f(\theta) + f\big((\pi - \theta)\big)$ for singlet state ($s = 0$) and $f(\theta) - f\big((\pi - \theta)\big)$ for triplet state ($s = 1$).

The statistical weight are: $2s + 1 = 1, for\ s = 0,$ and $2s + 1 = 3, for\ s = 1.$

We can write the Coulomb scattering amplitude $f_c(\theta)$ as:

$$f_c(\theta) = \frac{\eta}{2k\, sin^2\left(\frac{\theta}{2}\right)} \exp\left\{-i\eta \ln sin^2\left(\frac{\theta}{2}\right)\right\} \quad \ldots (6a)$$

$$f_c(\pi - \theta) = \frac{\eta}{2k\, cos^2\left(\frac{\theta}{2}\right)} \exp\left\{-i\eta \ln cos^2\left(\frac{\theta}{2}\right)\right\} \quad \ldots (6b)$$

As both the incident protons and the protons of target material have their spins randomly oriented, so the scattering cross-section for p-p scattering may be obtained after taking into account the proper statistical weights of the singlet and triplet cases, as given by

$$\begin{aligned}\sigma_c(\theta) &= \frac{1}{4}|f_s(\theta)|^2 + \frac{3}{4}|f_t(\pi - \theta)|^2 \\ &= \frac{1}{4}|f_c(\theta) + f_c(\pi - \theta)|^2 + \frac{3}{4}|f_c(\theta) - f_c(\pi - \theta)|^2 \\ &= |f_c(\theta)|^2 + |f_c(\pi - \theta)|^2 - Re\left[f^*{}_c(\theta). f_c(\pi - \theta)\right] \quad \ldots (7)\end{aligned}$$

$$so, \quad \sigma_c(\theta) = \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \left[\frac{1}{sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{cos^4\left(\frac{\theta}{2}\right)} - \frac{\cos\left(\eta \ln tan^2\left(\frac{\theta}{2}\right)\right)}{sin^2\left(\frac{\theta}{2}\right).cos^2\left(\frac{\theta}{2}\right)}\right] \quad \ldots (8)$$

The equation (8) is known as the **Mott scattering formula**. In equation (7), the **first term** describes **scattering**, **second term exchange scattering** and **third term** is **the interference between direct and exchange waves**.

For proton's energy (E) of 1 MeV and higher, the term $\eta = \frac{e^2}{4\pi\epsilon_0 \hbar v}$ is so small that the cosine in the last term in equation (8) is ≈ 1 unless $\theta = 0\ or\ \pi$. Therefore, equation (8) can be written as

$$\begin{aligned}\sigma_c(\theta) &= \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \left[\frac{1}{sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{cos^4\left(\frac{\theta}{2}\right)} - \frac{1}{sin^2\left(\frac{\theta}{2}\right).cos^2\left(\frac{\theta}{2}\right)}\right] \quad \ldots (9) \\ &= \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . [16 + 16 - 4 \times 4], \qquad for\ \theta = 90^\circ \\ &= 16 \times \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2\end{aligned}$$

Again, from classical treatment, we have the equation (4) for cross-section in the C-system is

$$\begin{aligned}\sigma_c(\theta) &= \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . \left[\frac{1}{sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{cos^4\left(\frac{\theta}{2}\right)}\right] \\ &= \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2 . [16 + 16], \qquad for\ \theta = 90^\circ \\ &= 2 \times 16 \times \left(\frac{e^2}{4\pi\epsilon_0 M v^2}\right)^2\end{aligned}$$

Therefore, the Mott formula gives a cross-section at $\theta = 90^\circ$ in the C-system (i.e., $\theta_L = \frac{\theta}{2} = 45^\circ$ in the L-system), which is **half** of what would be obtained if the effect of quantum mechanical indistinguishability is not taken into account. This is in **good agreement with observations at low energies**.

Effect of nuclear force in p-p scattering:

It is found that when the experimental results of p-p scattering are compared with those calculated from the Mott formula, there is **good agreement** only at very low energies protons ($\boldsymbol{E < 0.1\, MeV}$). But, for higher energies, the experimental values of the cross-section are found to differ significantly from the theoretical values.

For examples, for E upto about 0.6 MeV, the experimental values are lower; for $E > 0.6\, MeV$, they are higher than the theoretical values.

Such disagreement indicates that nuclear potential between the two protons must be taken into consideration at higher energies when the protons approach very close together. So, for the short-range nuclear force and low energies protons, only the $l = 0$ or S-scattering is expected to take place under the action of the nuclear potential.

If we write the wave function $\psi(r, \theta)$ in terms of partial waves of different orders, then we have **for pure Coulomb force**

$$\psi(r,\theta) = \frac{1}{r}\sum_{l=0}^{\infty} v_l(r)\, P_l(\cos\theta)$$

$$= \frac{v_0}{r} + \frac{1}{r}\sum_{l=1}^{\infty} v_l(r)\, P_l(\cos\theta), \quad since \;\; P_0(\cos\theta) = 1 \qquad \dots(10)$$

For **Coulomb plus nuclear forces**

$$\chi(r,\theta) = \frac{1}{r}\sum_{l=0}^{\infty} u_l(r)\, P_l(\cos\theta)$$

$$= \frac{u_0}{r} + \frac{1}{r}\sum_{l=1}^{\infty} u_l(r)\, P_l(\cos\theta) \qquad \dots(11)$$

Now, for $l > 0$, the nuclear force has no effect, because protons in higher orbital angular momentum states (p, d, f, … states) stay apart from each other beyond the range (b) of the nuclear force. Thus, for higher values of l, we must put

$$u_l(r) = v_l(r)$$

Hence, the expression (11) becomes

$$\chi(r,\theta) = \frac{u_0}{r} + \frac{1}{r}\sum_{l=1}^{\infty} v_l(r)\, P_l(\cos\theta)$$

$$= \frac{u_0}{r} + \psi(r,\theta) - \frac{v_0}{r}$$

$$= \psi(r,\theta) + \frac{u_0(r) - v_0(r)}{r}$$

$$= \psi(r,\theta) + \frac{\delta_0(r)}{r}$$

where, $\delta_0(r) = u_0(r) - v_0(r)$, called the phase factor. That is for $l = 0$, the radial function $u_0(r)$ for the combined Coulomb plus nuclear force differs from $v_0(r)$ by a phase factor of $\delta_0(r)$, and hence it is possible to normalize the functions $u_0(r)$ and $v_0(r)$.

It is then found that

$$\chi(r,\theta) = \exp i\{kz + \eta \ln k(r - z)\} + \frac{g(\theta)}{r}\exp\{i(kr - \eta \ln 2kr + 2\eta_0 + \pi)\} \qquad \dots(12)$$

Where,

$$g(\theta) = \frac{e^2}{4\pi\epsilon_0 Mv^2} \cdot \frac{\exp\left\{-i\eta \ln sin^2\left(\frac{\theta}{2}\right)\right\}}{sin^2\left(\frac{\theta}{2}\right)} + \frac{i}{2k}\{\exp(2i\delta_0) - 1\} \quad \dots (13)$$

Like in the case of pure Coulomb scattering, we have to make symmetric and antisymmetric combinations of the space functions $g(\theta)$ and $g(\pi - \theta)$, and multiply them by the appropriate statistical weights for spin orientations. This is given by

$$g_s(\theta) = g(\theta) + g(\pi - \theta)$$

$$= \left(\frac{e^2}{4\pi\epsilon_0 Mv^2}\right) \cdot \left\{\frac{\exp\left\{-i\eta \ln sin^2\left(\frac{\theta}{2}\right)\right\}}{sin^2\left(\frac{\theta}{2}\right)} + \frac{\exp\left\{-i\eta \ln cos^2\left(\frac{\theta}{2}\right)\right\}}{cos^2\left(\frac{\theta}{2}\right)} + \frac{i}{k}\{\exp(2i\delta_0) - 1\}\right\}$$

and

$$g_a(\theta) = g(\theta) - g(\pi - \theta)$$

$$= \left(\frac{e^2}{4\pi\epsilon_0 Mv^2}\right) \cdot \left\{\frac{\exp\left\{-i\eta \ln sin^2\left(\frac{\theta}{2}\right)\right\}}{sin^2\left(\frac{\theta}{2}\right)} + \frac{\exp\left\{-i\eta \ln cos^2\left(\frac{\theta}{2}\right)\right\}}{cos^2\left(\frac{\theta}{2}\right)}\right\}$$

It may be noted that the nuclear force does not contribute to the antisymmetric function for which the minimum l is 1. We get finally,

$$\sigma(\theta) = \frac{1}{4}|g_s(\theta)|^2 + \frac{3}{4}|g_a(\theta)|^2$$

$$= \frac{1}{4}|g(\theta) + g(\pi - \theta)|^2 + \frac{3}{4}|g(\theta) - g(\pi - \theta)|^2$$

$$so, \quad \sigma_c(\theta) = \left(\frac{e^2}{4\pi\epsilon_0 Mv^2}\right)^2 \cdot \left[\frac{1}{sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{cos^4\left(\frac{\theta}{2}\right)} - \frac{1}{sin^2\left(\frac{\theta}{2}\right) . cos^2\left(\frac{\theta}{2}\right)}\right]$$

$$+ \frac{sin^2\delta_0}{k^2} - \frac{e^2}{4\pi\epsilon_0 Mv^2} \cdot \frac{\sin\delta_0 . \cos\delta_0}{k . sin^2\left(\frac{\theta}{2}\right) . cos^2\left(\frac{\theta}{2}\right)} \quad \dots (14)$$

The **first term** in equation (14) is **Mott scattering formula** for Coulomb force only, the **second term** is the **nuclear scattering** and **last** represents the interference between the **Coulomb and nuclear scattering**.

The equation (4) reduces to equation (9) for a Coulomb force when the nuclear phase-shift $\delta_0 = 0$, i.e., there is no nuclear scattering effect. The second term will stand only, if the protons are unchanged. The sign of the last term depends on the sign of δ_0 and hence on the Fermi scattering length.

So, it is possible to determine the sign of δ_0 from the observed variation of the p-p scattering cross-section $\sigma_{pp}(\theta)$ with θ and to decide whether there is a bound state of the p-p system or not. The results show that the p-p scattering length is $-ve$ so that there cannot be any bound state of the p-p system. Further, the $+ve$ sign of δ_0 also shows that the p-p nuclear potential is attractive.

If a rectangular potential well is assumed for the nuclear part of the p-p potential, we found the depth $(V_o)_{pp} = 13.3\ MeV$ and range $b_{pp} = 2.58\ fm$. These values are fair agreement with the corresponding values of the singlet n-p potential of depth $V_{os} = 14.3\ MeV$ and range $b_s = 2.50\ fm$.

It may be noted that the spins of the protons must be antiparallel in the S-state to satisfy Pauli's principle which results in ${}^1_0S$ state for the p-p system.

As in the case of n-p scattering, low energy p-p scattering upto about $10\ MeV$ can be accounted for an S-wave ($l = 0$) interaction between the protons. Except in the forward and backward directions, the scattering is almost spherically symmetric.