Lecture 2: Concept of Classical Mechanics (continues)
Paper – PHS 204 (CBCS)
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Conservation principles (laws):

The word 'conservation' applies in the sense of constantness when some characteristics of the motion of a system remain constant in time. In our physical world there exist a number of conservation principles or laws, some exact and some approximate. There are different conservation laws that have been established as result of extensive research on particle systems.

Conservation of linear momentum:

In a closed system the total momentum is constant. This fact, known as the *law of conservation of momentum*, is implied by Newton's laws of motion. Suppose two particles of masses m_1 and m_2 interact to each other. Because of the Newton's third law of motion, the forces between them are equal and opposite. The Newton's second law states that $F_1 = \frac{dp_1}{dt}$ and $F_2 = \frac{dp_2}{dt}$. Therefore,

$$\frac{dp_1}{dt} = -\frac{dp_2}{dt}$$

with the negative sign indicating that the forces oppose. Equivalently,

$$\frac{d}{dt}(p_1 + p_2) = 0$$

This means the time-rate of change of total momentum is $= 0$, i.e., the applied force $F = 0$. From the above expression, it is clear that

$$p_1 + p_2 = constant$$

So, if the velocities of the particles are u_1 and u_2 before the interaction, and afterwards they are v_1 and v_2, then

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

That is, if the total force $F = 0$, then $\frac{dp}{dt} = 0$ and the linear momentum is conserved. This is the principle of conservation of **momentum**. Similarly, if there are several particles, the momentum exchanged between each pair of particles adds up to zero, so the total change in momentum is zero. The principle of conservation of **momentum** states that in an isolated system, two objects that collide have the same combined **momentum** before and after the collision.

Conservation of angular momentum:

We consider a particle of mass m and linear momentum p and velocity $\boldsymbol{v}$ at a position $\boldsymbol{r}$ relative to origin O of an inertial reference frame (Fig. 2). The a**ngular momentum** $\boldsymbol{L}$ of the particle with respect to the origin O to be

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \qquad \text{... (1)}$$

and the torque τ as the moment of the force about the origin O. i.e.,

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} \qquad \ldots (2)$$

where, $\boldsymbol{F}$ is the applied force opn the particle.

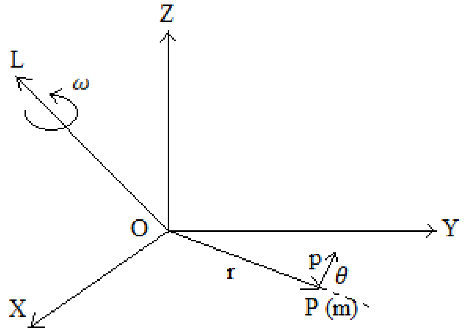

Fig. 2: Linear and angular momenta

Let us form the vector product of $\boldsymbol{\tau}$ with both sides of equation

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}(m\boldsymbol{v}) \qquad \ldots (3)$$

We can write,

$$\boldsymbol{\tau} = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}$$

$$= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) - \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p}$$

$$= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) - \boldsymbol{v} \times m\boldsymbol{v}$$

In the above expression, the second term is zero, as both vectors are parallel and therefore,

$$\boldsymbol{\tau} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \frac{d\boldsymbol{L}}{dt} \qquad \ldots (4)$$

Thus the time-rate of change of the vector angular momentum of a particle is equal to the vector torque acting on it. Equation (4) is the analogue of Newton's second law of motion in the case of rotational motion.

If $\tau = 0$, then

$$\frac{d\boldsymbol{L}}{dt} = 0, \qquad i.e., \qquad L = constant$$

So, the principle of the conservation of angular momentum states that the angular momentum is conserved in the absence of external torque.

Conservation of energy:

The work–energy theorem states that for a particle of constant mass *m*, the total work *W* done on the particle as it moves from position $\mathbf{r}_1$ to $\mathbf{r}_2$ is equal to the change in kinetic energy E_k of the particle:

$$W = \Delta E_k = E_{k,2} - E_{k,1} = \frac{1}{2}m({v_2}^2 - {v_1}^2)$$

Conservative forces can be expressed as the gradient of a scalar function, known as the potential energy and denoted E_p:

$$F = -\nabla E_p$$

If all the forces acting on a particle are conservative, and E_p is the total potential energy (which is defined as a work of involved forces to rearrange mutual positions of bodies), obtained by summing the potential energies corresponding to each force

$$F.\Delta r = -\nabla E_p.\Delta r = -\Delta E_p$$

The decrease in the potential energy is equal to the increase in the kinetic energy

$$-\Delta E_p = \Delta E_k \implies \Delta(E_k + E_p) = 0$$

This result is known as *conservation of energy* and states that the total energy,

$$\sum E = E_k + E_p$$

is constant in time. It is often useful, because many commonly encountered forces are conservative.

Newtonian mechanics:

The linear momentum of a rigid object of mass m is $= m\,\frac{dx}{dt}$, where $\boldsymbol{v} = \frac{dx}{dt}$ is the velocity of the object moving in the direction of the unit vector $\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$. Time is measured in units of seconds (s), and distance is measured in units of meters (m). The magnitude of momentum is measured in units of kilogram meters per second $(kg.m.s^{-1})$, and the magnitude of velocity (speed) is measured in units of meters per second $(m.s^{-1})$. Classical Newtonian mechanics assumes that there exists an inertial frame of reference for which the motion of the object is described by the differential equation

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = m\frac{d^2\boldsymbol{x}}{dt^2} \qquad \dots (1)$$

where the vector $\boldsymbol{F}$ is the force. The magnitude of force is measured in units of $newtons$ (N). The force appearing in equation (1) is a vector field. What this means is that the particle can be subject to a force, the magnitude and direction of which are different in different parts of space.

A measure of the forces experienced by a particle moving from position $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ in space is work. The work done moving the object from point 1 to point 2 in space along a path is defined as

$$W_{12} \equiv \int_{\boldsymbol{r}=\boldsymbol{r}_1}^{\boldsymbol{r}=\boldsymbol{r}_2} \boldsymbol{F}.d\boldsymbol{r} \qquad \dots (2)$$

where $\boldsymbol{r}$ is a spatial-position vector coordinate of the particle. Fig. 1 illustrates one possible trajectory for a particle moving from position $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$.

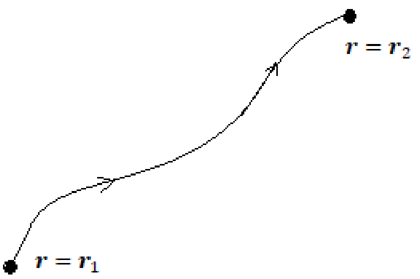

Fig. 1: Illustration of a classical particle trajectory from position $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ in space.

The definition of work is simply the integral of the force applied multiplied by the infinitesimal distance moved in the direction of the force for the complete path from point 1 to point 2. For a conservative force-field, the work W_{12} is the same for any path between points 1 and 2. Hence, making use of the fact $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt}$, the work is

$$W_{12} = \int_{r=r_1}^{r=r_2} \boldsymbol{F}.d\boldsymbol{r} = m\int \frac{d\boldsymbol{v}}{dt}.\boldsymbol{v}\,dt = \frac{m}{2}\int \frac{d}{dt}(v^2)\,dt \qquad ...(3)$$

so that $W_{12} = m\,({v_2}^2 - {v_1}^2)/2 = T_2 - T_1$, $v^2 = \boldsymbol{v}.\boldsymbol{v}$, and the scalar $T = mv^2/2$ is called the kinetic energy of the object.

For conservative forces, because the work done is the same for any path between points 1 and 2, the work done around any closed path, such as the one illustrated in Fig. 2, is always zero, or

$$\oint \boldsymbol{F}.d\boldsymbol{r} = 0 \qquad ...(4)$$

This fact remains true if force is the gradient of a single-valued spatial scalar field where

$$F = -\nabla V(\boldsymbol{r}) \qquad ...(5)$$

since $\oint \boldsymbol{F}.d\boldsymbol{r} = -\oint \nabla V.d\boldsymbol{r} = -\oint dV = 0$. In equation (5), $V(\boldsymbol{r})$ is the potential energy and is measured in joules (J) or electron volts (eV). If the forces acting on the object are conservative, then total energy, which is the sum of kinetic and potential energy, is a constant of the motion. In other words, total energy $T + V$ is conserved.

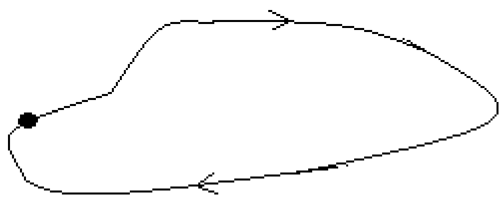

Fig. 2: Illustration of a closed-path classical particle trajectory.

Because kinetic and potential energy can be expressed as functions of the particle's position and time, it is possible to define a Hamiltonian function H for the system, which is

$$H = T + V \qquad ...(6)$$

The Hamiltonian function may then be used to describe the dynamics of particles in the system.

For a non-conservative force, such as a particle subject to frictional forces, the work done around any closed path is not zero, and $\oint \boldsymbol{F}.d\boldsymbol{r} \neq 0$.

The concept of force has been introduced to help ensure that the motion of objects can be described as a simple process of cause and effect. A force-field in three dimensional space is assumed to exist and is represented mathematically as a continuous, integrable vector field, $\boldsymbol{F}(r)$. If time is also continuous and integrable, a conservative force-field energy is conveniently partitioned between a kinetic and potential term and total energy is conserved. By simply representing the total energy as a function or Hamiltonian, $H = T + V$, a differential equation that describes the dynamics of the object can be found. Integration of the differential equation of motion gives the trajectory of the object as it moves through space.

In practice, these ideas are very powerful and may be applied to many problems involving the motion of macroscopic objects. As an example, consider the basic problem of finding the motion of a particle mass, m, attached to a spring. Of course, the solution will be oscillatory and so characterized by a frequency and amplitude of oscillation. However, the power of the theory is that relationships among all the parameters that govern behavior of the system may be obtained.

Importance of classical mechanics:

Classical mechanics is a theory useful for the study of the motion of non-quantum mechanical, low-energy particles in weak gravitational fields. Also, it has been extended into the complex domain where complex **classical mechanics** exhibits behaviors very similar to quantum **mechanics.** Classical mechanics accurately describes the behavior of most "normal" objects.

Difficulties in classical mechanics:

Classical mechanics is unable to explain many phenomena, which often showed a constraint pattern. Some difficulties were discovered in the late 19th century that could only be resolved by more modern physics. Particularly, the difficulty arose when scienctists tried to probe into the mysteries of smallest constituents of matter and systems involving them, namely, the atoms and the molecules. The different behaviour of classical electromagnetism and classical mechanics under velocity transformations led to the special theory of relativity, often included in the term classical mechanics. A second set of difficulties were related to thermodynamics. When combined with thermodynamics, classical mechanics leads to the Gibbs paradox of classical statistical mechanics, in which entropy is not a well-defined quantity. Black-body radiation was not explained without the introduction of quanta. As experiments reached the atomic level, the classical mechanics has been failed to explain, even approximately, such basic things as the energy levels and sizes of atoms and the photo-electric effect. The effort at resolving these problems led to the development of quantum mechanics.

Some questions and answers:

Example 1: What is classical mechanics?

Classical mechanics is the study of the *motion* of bodies (including the special case in which bodies remain at rest) in accordance with the general principles first enunciated by Sir Isaac Newton in his *Philosophiae Naturalis Principia Mathematica* (1687), commonly known as the *Principia*. Classical mechanics was the first branch of Physics to be discovered, and is the foundation upon which all other branches of Physics are built. Moreover, classical mechanics has many important applications in other areas of science, such as Astronomy (*e.g.,* celestial mechanics), Chemistry (*e.g.,* the dynamics of molecular collisions), Geology (*e.g.,* the propagation of seismic waves, generated by earthquakes, through the Earth's crust), and Engineering (*e.g.,* the equilibrium and stability of structures). Classical mechanics is also of great significance outside the realm of science. After all, the sequence of events leading to the discovery of classical mechanics--starting with the ground-breaking work of Copernicus, continuing with the researches of Galileo, Kepler, and Descartes, and culminating in the monumental achievements of Newton--involved the complete overthrow of the Aristotelian picture of the Universe, which had previously prevailed for more than a millennium, and its replacement by a recognizably modern picture in which humankind no longer played a privileged role.

In our investigation of classical mechanics we shall study many different types of motion, including:

Translational motion--motion by which a body shifts from one point in space to another (*e.g.*, the motion of a bullet fired from a gun).

Rotational motion--motion by which an extended body changes orientation, with respect to other bodies in space, without changing position (*e.g.*, the motion of a spinning top).

Oscillatory motion--motion which continually repeats in time with a fixed period (*e.g.*, the motion of a pendulum in a grandfather clock).

Circular motion--motion by which a body executes a circular orbit about another fixed body [*e.g.*, the (approximate) motion of the Earth about the Sun].

Of course, these different types of motion can be combined: for instance, the motion of a properly bowled bowling ball consists of a combination of translational and rotational motion, whereas wave propagation is a combination of translational and oscillatory motion. Furthermore, the above mentioned types of motion are not entirely distinct: *e.g.*, circular motion contains elements of both rotational and oscillatory motion. We shall also study *statics*: *i.e.*, the subdivision of mechanics which is concerned with the forces that act on bodies *at rest* and in equilibrium. Statics is obviously of great importance in civil engineering: for instance, the principles of statics were used to design the building in which this lecture is taking place, so as to ensure that it does not collapse.

Example 2: Mr. X achieved notoriety by allegedly jumping off a bridge, for a bet, and surviving. Given that the bridge rises $135ft$ over a river, how long would Mr. X have been in the air, and with what speed would he have struck the water ? You may neglect air resistance.

Answer: Mr. X's net vertical displacement was $h = -135\ ft = -135 \times 0.3048\ m \approx 41.15\ m$. Assuming his initial velocity was zero. We can write

$$h = -\frac{1}{2}gt^2$$

where t was his time of flight, and $g = 9.81\ m.s^{-2}$. Hence,

$$t = \sqrt{-\frac{2h}{g}} = \sqrt{\frac{2 \times 41.15}{9.81}} \approx 2.896\ s$$

His final velocity was

$$v = -gt = -9.81 \times 2.896\ \approx -28.41\ m.s^{-1}$$

Thus, the speed with which he plunged into the river was $28.41\ m.s^{-1}$.

Example 3: What is the inadequacy of classical mechanics?

Answer: The inadequacies of classical mechanics are as follows:

(i) It does not hold in the area of atomic dimensions.
(i) This could not clarify the observed spectra of black body radiation.
(ii) This could not clarify the observed variation of specific heat of metals and gases.
(iii) It could not clarify the inception of discrete spectra of molecules, since as per classical mechanics; the vitality changes are constantly consistent.
(iv) Classical mechanics could not clarify a large number of phenomena, such as photoelectric effect, Raman Effect, and so on.

References:

1. Aydin Sayili (1987). "Ibn Sīnā and Buridan on the Motion of the Projectile". *Annals of the New York Academy of Sciences*. **500** (1): 477–482. Bibcode:1987NYASA.500..477S. doi:10.1111/j.1749-6632.1987.tb37219.x.
2. A. F. J. Levi (2016). Chapter 1, Concepts in classical mechanics, pp 1-1 to 1-11. Essential Classical Mechanics for Device Physics. IOP Concise Physics.
3. .History of classical mechanics. https://en.wikipedia.org/wiki/History_of_classical_mechanics.
4. Rovelli, Carlo (2015). "Aristotle's Physics: A Physicist's Look". *Journal of the American Philosophical Association*. **1** (1): 23–40. arXiv:1312.4057. doi:10.1017/apa.2014.11.
5. Peter Pesic (March 1999). "Wrestling with Proteus: Francis Bacon and the "Torture" of Nature". *Isis*. The University of Chicago Press on behalf of The History of Science Society. **90** (1): 81–94. doi:10.1086/384242. JSTOR 237475.
6. S. L. Gupta, V. Kumar and H. V. Sharma (2000). Classical mechanics. Pragati Prakashan, Post Box No. 62, Begun Bridge, Meerut – 250001, India.
7. McGill and King (1995). *Engineering Mechanics, An Introduction to Dynamics* (3rd ed.). PWS Publishing Company. ISBN 978-0-534-93399-9.