**M.Sc. Course in**

**Applied Mathematics**

**with Oceanology and Computer Programming**

**Vidyasagar University**

Semester-II

**Paper-MTM 202** **Paper Name: Numerical Analysis**

# Solution of System of Linear Equations

## Module No. 3

# Solution of Tri-diagonal System of Equations

***Objective***

(a) Evaluation of Tri-diagonal determinant

(b) Solution of tri-diagonal system of equations

***Keywords***

band matrix, tri-diagonal determinant, tri-diagonal equations

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In this module, a special type of system of equations called tri-diagonal system of equations is introduced here. The tri-diagonal system occurs in many applications. A special case of LU-decomposition method is used to solve a tri-diagonal system off equations.

## 3.1 Solution of tri-diagonal systems

The tri-diagonal system of equations is a particular case of a system of linear equations. These type of equations occur in many applications, viz. cubic spline interpolation, solution of boundary value problem, etc. The tri-diagonal system of equations is of the following form

$$\begin{aligned} b_1x_1 + c_1x_2 &= d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 &= d_2 \\ a_3x_2 + b_3x_3 + c_3x_4 &= d_3 \\ \cdots\cdots\cdots\cdots\cdots &\quad \cdots \\ a_nx_{n-1} + b_nx_n &= d_n. \end{aligned} \tag{3.1}$$

The coefficient matrix for this system is

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_n & b_n \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \tag{3.2}$$

This matrix has many interesting properties. Note that the main diagonal and its two adjacent (below and upper) diagonals are non-zero and all other elements are zero. This special matrix is called tri-diagonal matrix and the system of equations is called a **tri-diagonal system** of equations. This matrix is also known as band matrix.

A tri-diagonal system of equations can be solved by the methods discussed earlier. But, this system has some special properties. Exploring these special properties, the system can be solved by a simple way, starting from the **LU** decomposition method.

Let $\mathbf{A} = \mathbf{LU}$ where

$$\mathbf{L} = \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_{n-1} & \gamma_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta_n & \gamma_n \end{bmatrix},$$

$$\text{and } \mathbf{U} = \begin{bmatrix} 1 & \alpha_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \alpha_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \alpha_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{LU} = \begin{bmatrix} \gamma_1 & \gamma_1\alpha_1 & 0 & \cdots & 0 & 0 & 0 \\ \beta_2 & \alpha_1\beta_2 + \gamma_2 & \alpha_2\gamma_2 & \cdots & 0 & 0 & 0 \\ 0 & \beta_3 & \alpha_2\beta_3 + \gamma_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_n & \beta_n\alpha_{n-1} + \gamma_n \end{bmatrix}.$$

Now, comparing both sides of the matrix equation $\mathbf{LU} = \mathbf{A}$ and we obtain the following system of equations.

$$\begin{aligned} &\gamma_1 = b_1, \qquad \gamma_i\alpha_i = c_i, \qquad \text{or,} \quad \alpha_i = c_i/\gamma_i, \qquad i = 1, 2, \ldots, n-1 \\ &\beta_i = a_i, \quad i = 2, \ldots, n \\ &\gamma_i = b_i - \alpha_{i-1}\beta_i = b_i - a_i\frac{c_{i-1}}{\gamma_{i-1}}, \qquad i = 2, 3, \ldots, n. \end{aligned}$$

Hence, the elements of the matrices $\mathbf{L}$ and $\mathbf{U}$ are given by the following equations.

$$\gamma_1 = b_1,$$

$$\gamma_i = b_i - \frac{a_i c_{i-1}}{\gamma_{i-1}}, \quad i = 2, 3, \ldots, n \tag{3.3}$$

$$\beta_i = a_i, \quad i = 2, 3, \ldots, n \tag{3.4}$$

$$\alpha_i = c_i/\gamma_i, \quad i = 1, 2, \ldots, n-1. \tag{3.5}$$

Note that, this is a very simple system of equations.

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Now, the solution of the equation $\mathbf{Ax} = \mathbf{d}$ where $\mathbf{d} = (d_1, d_2, \ldots, d_n)^t$ can be obtained by solving the equation $\mathbf{Lz} = \mathbf{d}$ by forward substitution and then by solving the equation $\mathbf{Ux} = \mathbf{z}$ by back substitution.

The solution of the equation $\mathbf{Lz} = \mathbf{d}$ is

$$z_1 = \frac{d_1}{b_1}, \quad z_i = \frac{d_i - a_i z_{i-1}}{\gamma_i}, \quad i = 2, 3, \ldots, n. \tag{3.6}$$

And the solution of the equation $\mathbf{Ux} = \mathbf{z}$ is

$$x_n = z_n, \quad x_i = z_i - \alpha_i x_{i+1} = z_i - \frac{c_i}{\gamma_i} x_{i+1}, \qquad i = n-1, n-2, \ldots, 1. \tag{3.7}$$

Observe that the number of computations is linear, i.e. $O(n)$ for $n$ equations. Thus, this special method needs significantly less time compare to other method to solve a tri-diagonal equations.

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**Example 3.1** *Solve the following tri-diagonal system of equation*

$x_1 + 2x_2 \qquad = 4, \qquad -x_1 + 2x_2 + 3x_3 = 6, \qquad 3x_2 + x_3 = 8.$

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**Solution.** For this problem, $b_1 = 1, c_1 = 2, a_2 = -1, b_2 = 2, c_2 = 3, a_3 = 3, b_3 = 1,$ $d_1 = 4, d_2 = 6, d_3 = 8.$

Thus,

$$\begin{aligned}
&\gamma_1 = b_1 = 1 \\
&\gamma_2 = b_2 - a_2 \frac{c_1}{\gamma_1} = 2 - (-1).2 = 4 \\
&\gamma_3 = b_3 - a_3 \frac{c_2}{\gamma_2} = 1 - 3.\frac{3}{4} = -\frac{5}{4} \\
&z_1 = \frac{d_1}{b_1} = 4, \qquad z_2 = \frac{d_2 - a_2 z_1}{\gamma_2} = \frac{5}{2}, \qquad z_3 = \frac{d_3 - a_3 z_2}{\gamma_3} = -\frac{2}{5} \\
&x_3 = z_3 = -\frac{2}{5}, \qquad x_2 = z_2 - \frac{c_2}{\gamma_2} x_3 = \frac{14}{5}, \qquad x_1 = z_1 - \frac{c_1}{\gamma_1} x_2 = -\frac{8}{5}.
\end{aligned}$$

Therefore, the required solution is $x_1 = -\frac{8}{5}, x_2 = \frac{14}{5}, x_3 = -\frac{2}{5}$.

The above method is not applicable for all kinds of tri-diagonal equations. The equations (3.6) and (3.7) are valid only if $\gamma_i \neq 0$ for all $i = 1, 2, \ldots, n$. If any one of $\gamma_i$ is zero at any stage, then the method fails. Remember that this method is based on LU decomposition method and LU decomposition method is applicable and gives unique solution if all the principal minors of the coefficient matrix are non-zero. Fortunately, a modified method is available if one or more $\gamma_i$ are zero. The modified method is described below.

Without loss of generality, let us assume that $\gamma_k = 0$ and $\gamma_i \neq 0, i = 1, 2, \ldots, k-1$. Let $\gamma_k = x$, $x$ is a symbolic value of $\gamma_k$. The values of the other $\gamma_i, i = k+1, \ldots, n$ are calculated by using the equation (3.3). Using these $\gamma$'s, the values of $z_i$ and $x_i$ are determined by the formulae (3.6) and (3.7). Note that, in general, the values of $x_i$'s are depend on $x$. Practically, the value of $x$ is 0. Finally, the solution is obtained by substituting $x = 0$.

## 3.2 Evaluation of tri-diagonal determinant

For $n \geq 3$, the general form of a tri-diagonal matrix $\mathbf{T} = [t_{ij}]_{n\times n}$ is

$$\mathbf{T} = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & \cdots & \cdots & \cdots & 0 \\ 0 & a_3 & b_3 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix}$$

and $t_{ij} = 0$ for $|i - j| \geq 2$.

Note that the first and last rows contain two non-zero elements and the number of non-zero elements to all other rows is three. These elements may also be zero for any particular case.

In general, for any matrix the number of non-zero elements are $n^2$, but for tri-diagonal matrix only $3(n-2)+4 = 3n-2$ non-zero elements are present. So, this matrix can be stored using only three vectors $\mathbf{c} = (c_1, c_2, \ldots, c_{n-1})$, $\mathbf{a} = (a_2, a_3, \ldots, a_n)$, and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$.

Let us define a vector $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ as

$$d_i = \begin{cases} b_1 & \text{if } i = 1 \\ b_i - \dfrac{a_i}{d_{i-1}} c_{i-1} & \text{if } i = 2, 3, \ldots, n. \end{cases} \tag{3.8}$$

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The value of the determinant is the product $P = \prod_{i=1}^{n} d_i$.

If $d_i = 0$ for any particular $i$, $i = 1, 2, \ldots, n$, then, set $d_i = x$ ($x$ is just a symbolic name). Using this value calculate other $d$'s, i.e. $d_{i+1}, d_{i+2}, \ldots, d_n$. In this case, $d$'s contains $x$. Thus, the product $P = \prod_{i=1}^{n} d_i$ depends on $x$. Lastly, the value of the determinant is obtained by substituting $x = 0$ in $P$.

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**Example 3.2** *Find the values of the following tri-diagonal determinants.*

$$\mathbf{A} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & -3 \\ 0 & -1 & 3 \end{vmatrix}, \qquad \mathbf{B} = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{vmatrix}.$$

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**Solution.** For the determinant **A**.

$d_1 = 1$, $d_2 = b_2 - \frac{a_2}{d_1}c_1 = 0$. Here $d_2 = 0$, so let $d_2 = x$. $d_3 = b_3 - \frac{a_3}{d_2}c_2 = \frac{3x-3}{x}$.

Thus, $P = d_1 d_2 d_2 = 1 \cdot x \cdot \frac{3x-3}{x} = 3x - 3$.

Now, we put $x = 0$. Therefore, $\mathbf{A} = -3$.

For the determinant **B**.

$d_1 = 1, d_2 = b_2 - \frac{a_2}{d_1}c_1 = 4$, $d_3 = b_3 - \frac{a_3}{d_2}c_2 = \frac{3}{2}$.

Therefore, $P = d_1 d_2 d_2 = 1 \cdot 4 \cdot \frac{3}{2} = 6$, that is, $\mathbf{B} = 6$.

## Self Assessment (MCQ/Short answer questions)

1. The time complexity to solve a system of $n$ tri-diagonal linear equations is
   (a) $O(n^2)$ (b) $O(n)$ (c) $O(n^3)$ (d) none of these

2. The solution of the tri-diagonal system of equations $x_1+x_2 = 3$, $x_1+x_2-3x_3 = -3$, $-2x_2 + 3x_3 = 4$ is
   (a) $x_1 = 1, x_2 = 2, x_3 = 3$ (b) $x_1 = 1, x_2 = 1, x_3 = 2$
   (c) $x_1 = 2, x_2 = 1, x_3 = 2$ (d) $x_1 = 2, x_2 = 1, x_3 = 1$

3. The method to solve a tri-diagonal system of equations is based on LU-decomposition method.
   (a) true (b) false

4. The tri-diagonal determinant of order $n$ can be evaluated in $\cdots\cdots$ time.

## Answer to the questions

1. (b)
2. (c)
3. (a)
4. $O(n)$

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## Self Assessment (Long answer questions)

1. Find the solution of the following tri-diagonal system: $2x_1 - 2x_2 = 1$
   $x_1 + 2x_2 - 3x_3 = -2$
   $-2x_2 + 2x_3 - 4x_4 = -1$
   $x_3 - x_4 = 3.$

2. Consider the following tri-diagonal linear system and assume that the coefficient matrix is diagonally dominant.
   $d_1x_1 + c_1x_2 = b_1$
   $a_1x_1 + d_2x_2 + c_2x_3 = b_2$
   $a_2x_2 + d_3x_3 + c_4x_4 = b_3$
   $\cdots\cdots\cdots\cdots\cdots\cdots$
   $a_{n-2}x_{n-2} + d_{n-1}x_{n-1} + c_{n-1}x_n = b_{n-1}$
   $a_{n-1}x_{n-1} + d_nx_n = b_n.$

   Write an iterative algorithm that to solve this system.

## Learn More

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