M.Sc. Course in
Applied Mathematics
with Oceanology and Computer Programming
Vidyasagar University

Semester-II

Paper-MTM 202 **Paper Name: Numerical Analysis**

Solution of Non-Linear Equation

Module No. 2

Solution of System of Non-linear Equations

Objective

(a) System of non-linear equations

(b) Fixed point iteration method

(c) Seidal method

(d) Newton-Raphson method

Keywords

roots of nonlinear equations, Fixed point iteration, Seidal method, Newton-Raphson method, convergence

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In UG level we studied Newton-Raphson method to solve a nonlinear equation (algebraic or transcendental). Here only one equation is consider at a time. We observed that the solution of a nonlinear equation is a difficult task. The nonlinear equation $f(x) = 0$ represents a plane curve and if the equation cuts the x-axis at the point $(x_1, 0)$, then x_1 is the root of the equation. But, a pair of nonlinear equations $f(x, y) = 0, g(x, y) = 0$ represent surfaces in 3-dimension. A point of intersection of these surfaces is the solution of the equations. But, it is a very difficult task to find such point of intersection.

In this module, three methods, viz. fixed point iteration, Seidel iteration and Newton-Raphson method are discussed to solve a pair of nonlinear equations. These methods can also be extended to solve a system of three or more equations/variables.

2.1 Fixed point iteration method

Let a pair of nonlinear equations be

$$\begin{aligned} & f(x, y) = 0 \\ \text{and} \qquad & g(x, y) = 0. \end{aligned} \tag{2.1}$$

The equations $f(x, y) = 0$ and $g(x, y) = 0$ are either algebraic and/or transcendental. Like fixed point iteration, in case of single variable, these equations are rewritten as

$$\begin{aligned} & x = \phi(x, y) \\ \text{and} \qquad & y = \psi(x, y). \end{aligned} \tag{2.2}$$

The functions ϕ and ψ are not unique. The equations $f(x, y) = 0, g(x, y) = 0$ can be written in many different ways to get ϕ and ψ. All such representations are not acceptable. They must satisfy some conditions discussed latter.

For example, let $f(x, y) \equiv x^2y + 2x - 3y = 0$, $g(x, y) \equiv xy + e^x - y^2 = 0$. From first equation, $x = \frac{1}{2}(y - x^2y)$ or $x = \sqrt{\frac{3y-2x}{y}}$ and from second equation we can write $y = \sqrt{xy + e^x}$ or $y = \frac{1}{x}(y^2 - e^x)$.

Let (ξ, η) be an exact root of the pair of equations (2.2) and let (x_0, y_0) be the initial guess for this root.

The first approximate root (x_1, y_1) is then determine as $x_1 = \phi(x_0, y_0), y_1 = \psi(x_0, y_0)$. Similarly, the second approximate root is given by $x_2 = \phi(x_1, y_1), y_2 = \psi(x_1, y_2)$ and on. In general,

$$x_{n+1} = \phi(x_n, y_n),\ y_{n+1} = \psi(x_n, y_n). \tag{2.3}$$

Thus, the fixed point iteration method generates a (double) sequence of numbers $\{(x_n, y_n)\}$. If the sequence converges, i.e.

$$\lim_{n\to\infty} x_n = \xi \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \eta,$$

then

$$\xi = \phi(\xi, \eta) \qquad \text{and} \qquad \eta = \psi(\xi, \eta). \tag{2.4}$$

But, there is no guarantee that the sequence $\{(x_n, y_n)\}$ will converges to a root. The sufficient condition for convergent is stated below.

Theorem 2.1 *Let R be a region containing the root (ξ, η). Assumed that the functions $x = \phi(x, y), y = \psi(x, y)$ and their first order partial derivatives are continuous within the region R. If the initial guess (x_0, y_0) is sufficiently close to (ξ, η) and if*

$$\left|\frac{\partial\phi}{\partial x}\right| + \left|\frac{\partial\phi}{\partial y}\right| < 1 \qquad \textit{and} \qquad \left|\frac{\partial\psi}{\partial x}\right| + \left|\frac{\partial\psi}{\partial y}\right| < 1, \tag{2.5}$$

for all $(x, y) \in R$, then the sequence $\{(x_n, y_n)\}$ obtained from the equation (2.3) converges to the root (ξ, η).

In case of three variables the sufficient condition is stated below. The condition for the functions $x = \phi(x, y, z), y = \psi(x, y, z), z = \zeta(x, y, z)$ is

$$\begin{aligned} \left|\frac{\partial\phi}{\partial x}\right| + \left|\frac{\partial\phi}{\partial y}\right| + \left|\frac{\partial\phi}{\partial z}\right| &< 1, \\ \left|\frac{\partial\psi}{\partial x}\right| + \left|\frac{\partial\psi}{\partial y}\right| + \left|\frac{\partial\psi}{\partial z}\right| &< 1 \\ \text{and } \left|\frac{\partial\zeta}{\partial x}\right| + \left|\frac{\partial\zeta}{\partial y}\right| + \left|\frac{\partial\zeta}{\partial z}\right| &< 1 \end{aligned}$$

for all $(x, y, z) \in R$.

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Example 2.1 *Solve the following system of equations*

$$x^2 + y^2 - 4x = 0, \qquad x^2 + y^2 - 8x + 15 = 0$$

starting with $(3.5, 1.0)$ *by iteration method.*

Solution.

From first equation we have $x = 2+\sqrt{4-y^2}$ and from second equation $y = \sqrt{1-(x-4)^2}$.

Let, $\phi(x, y) = 2 + \sqrt{4-y^2}$ and $\psi(x, y) = \sqrt{1-(x-4)^2}$.

The iteration scheme is

$$x_{n+1} = \phi(x_n, y_n) = 2 + \sqrt{4-y_n^2},$$
$$\text{and } y_{n+1} = \psi(x_n, y_n) = \sqrt{1-(x_n-4)^2}.$$

The value of x_n, y_n, x_{n+1} and y_{n+1} for $n = 0, 1, \ldots$ are shown in the following table.

n	x_n	y_n	x_{n+1}	y_{n+1}
0	3.500000	1.000000	3.732051	0.866025
1	3.732051	0.866025	3.802776	0.963433
2	3.802776	0.963433	3.752654	0.980358
3	3.752654	0.980358	3.743243	0.968927
4	3.743243	0.968927	3.749623	0.966476
5	3.749623	0.966476	3.750978	0.968148
6	3.750978	0.968148	3.750054	0.968498
7	3.750054	0.968498	3.749861	0.968260
8	3.749861	0.968260	3.749992	0.968210
9	3.749992	0.968210	3.750020	0.968244
10	3.750020	0.968244	3.750001	0.968251
11	3.750001	0.968251	3.749997	0.968246

Thus, a root correct up to five decimal places is $(3.75000, 0.96825)$.

2.2 Seidal method

The above method can be accelerated by modified the iteration scheme (2.3). The modification is very simple. When y_{n+1} is computed, then the value of x_{n+1} is already available. So, we can use this updated value of x while calculating y_{n+1}. Thus, the modified iteration scheme is

$$\begin{aligned} x_{n+1} &= \phi(x_n, y_n) \\ y_{n+1} &= \psi(x_{n+1}, y_n). \end{aligned} \tag{2.6}$$

This method is called Seidal iteration.

In case of three variables this iteration scheme is extended as

$$\begin{aligned} x_{n+1} &= \phi(x_n, y_n, z_n) \\ y_{n+1} &= \psi(x_{n+1}, y_n, z_n) \\ \text{and } z_{n+1} &= \zeta(x_{n+1}, y_{n+1}, z_n). \end{aligned} \tag{2.7}$$

Example 2.2 *Solve the following system of equations*

$$x^2 + y^2 - 4x = 0, \qquad x^2 + y^2 - 8x + 15 = 0$$

starting with $(3.5, 1.0)$ *by Seidal iteration method.*

Solution.

The first equation can be written as $x = 2 + \sqrt{4 - y^2}$ and second equation be $y = \sqrt{1 - (x-4)^2}$.

Thus, $\phi(x, y) = 2 + \sqrt{4 - y^2}$ and $\psi(x, y) = \sqrt{1 - (x-4)^2}$.

The iteration scheme is

$$\begin{aligned} x_{n+1} &= \phi(x_n, y_n) = 2 + \sqrt{4 - y_n^2}, \\ \text{and } y_{n+1} &= \psi(x_{n+1}, y_n) = \sqrt{1 - (x_{n+1} - 4)^2}. \end{aligned}$$

The value of x_n, y_n, x_{n+1} and y_{n+1} for $n = 0, 1, \ldots$ are shown in the following table.

n	x_n	y_n	x_{n+1}	y_{n+1}
0	–	1.000000	3.500000	1.000000
1	3.500000	1.000000	3.732051	0.963433
2	3.732051	0.963433	3.752654	0.968927
3	3.752654	0.968927	3.749623	0.968148
4	3.749623	0.968148	3.750054	0.968260
5	3.750054	0.968260	3.749992	0.968244
6	3.749992	0.968244	3.750001	0.968246

Therefore, a root correct up to five decimal places is $(3.75000, 0.96825)$.

Observed that this problem has been solved in Example 2.1 and iteration method takes 11 iterations, while Seidal method takes only 6 iterations to obtained the same result.

2.3 Newton-Raphson method

Another efficient method to solve a pair of nonlinear equations is Newton-Raphson method. Let the pair of equations be

$$f(x, y) = 0 \text{ and } g(x, y) = 0. \tag{2.8}$$

Also, let (x_0, y_0) be an initial guess to the root (ξ, η). If h_0 and k_0 be the errors at x_0 and y_0 respectively, then $(x_0 + h_0, y_0 + k_0)$ is a root of the given equations.

Therefore,

$$\begin{aligned} f(x_0 + h_0, y_0 + k_0) &= 0 \\ g(x_0 + h_0, y_0 + k_0) &= 0. \end{aligned} \tag{2.9}$$

If $f(x, y)$ and $g(x, y)$ are differentiable, then by Taylor's series expansion, we have

$$\begin{aligned} f(x_0, y_0) + h_0 \left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + k_0 \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} + \cdots = 0 \\ g(x_0, y_0) + h_0 \left(\frac{\partial g}{\partial x}\right)_{(x_0, y_0)} + k_0 \left(\frac{\partial g}{\partial y}\right)_{(x_0, y_0)} + \cdots = 0 \end{aligned} \tag{2.10}$$

Neglecting square and higher order terms of h_0 and k_0, the equations of (2.10) reduce to

$$\begin{aligned} h_0\frac{\partial f_0}{\partial x} + k_0\frac{\partial f_0}{\partial y} &= -f_0 \\ h_0\frac{\partial g_0}{\partial x} + k_0\frac{\partial g_0}{\partial y} &= -g_0 \end{aligned} \tag{2.11}$$

where $f_0 = f(x_0, y_0)$, $\frac{\partial f_0}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)}$ etc.

The above equations are written as matrix notation shown below:

$$\begin{bmatrix} \frac{\partial f_0}{\partial x} & \frac{\partial f_0}{\partial y} \\ \frac{\partial g_0}{\partial x} & \frac{\partial g_0}{\partial y} \end{bmatrix} \begin{bmatrix} h_0 \\ k_0 \end{bmatrix} = \begin{bmatrix} -f_0 \\ -g_0 \end{bmatrix}.$$

This is a system of two equations and two variables. It can be solved by matrix inverse method or Crammer rule or any other method.

By Crammer rule, the values of h_0 and k_0 are obtained as

$$h_0 = \frac{1}{J_0}\begin{vmatrix} -f_0 & \frac{\partial f_0}{\partial y} \\ -g_0 & \frac{\partial g_0}{\partial y} \end{vmatrix}, \quad k_0 = \frac{1}{J_0}\begin{vmatrix} \frac{\partial f_0}{\partial x} & -f_0 \\ \frac{\partial g_0}{\partial x} & -g_0 \end{vmatrix}, \text{ where } J_0 = \begin{vmatrix} \frac{\partial f_0}{\partial x} & \frac{\partial f_0}{\partial y} \\ \frac{\partial g_0}{\partial x} & \frac{\partial g_0}{\partial y} \end{vmatrix}. \tag{2.12}$$

In this process the second and higher power of h_0 and k_0 are neglected, so the values of h_0 and k_0 obtained from equation (2.12) are approximate. Thus, $(x_0 + h_0, y_0 + k_0)$ is not an exact root, but it is more better root than the initial root (x_0, y_0). Let this new approximate root be (x_1, y_1), where

$$x_1 = x_0 + h_0, \qquad y_1 = y_0 + k_0. \tag{2.13}$$

Similarly, the second approximate root is $x_2 = x_1 + h_1, \qquad y_2 = y_1 + k_1$, where

$$h_1 = \frac{1}{J_1}\begin{vmatrix} -f_1 & \frac{\partial f_1}{\partial y} \\ -g_1 & \frac{\partial g_1}{\partial y} \end{vmatrix}, \quad k_1 = \frac{1}{J_1}\begin{vmatrix} \frac{\partial f_1}{\partial x} & -f_1 \\ \frac{\partial g_1}{\partial x} & -g_1 \end{vmatrix}, J_1 = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \end{vmatrix}.$$

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All the derivatives are calculated at the point (x_1, y_1).

In general, the $(n+1)$th approximate root (x_{n+1}, y_{n+1}) is given by

$$x_{n+1} = x_n + h_n, \qquad y_{n+1} = y_n + k_n, \tag{2.14}$$

where

$$h_n = \frac{1}{J_n}\begin{vmatrix} -f_n & \dfrac{\partial f_n}{\partial y} \\ -g_n & \dfrac{\partial g_n}{\partial y} \end{vmatrix}, \quad k_n = \frac{1}{J_n}\begin{vmatrix} \dfrac{\partial f_n}{\partial x} & -f_n \\ \dfrac{\partial g_n}{\partial x} & -g_n \end{vmatrix} \text{ and } J_n = \begin{vmatrix} \dfrac{\partial f_n}{\partial x} & \dfrac{\partial f_n}{\partial y} \\ \dfrac{\partial g_n}{\partial x} & \dfrac{\partial g_n}{\partial y} \end{vmatrix}. \tag{2.15}$$

Here also all derivatives are calculated at the point (x_n, y_n).

The method will terminate when $|x_{n+1} - x_n| < \varepsilon$ and $|y_{n+1} - y_n| < \varepsilon$, where ε is a very small positive pre-assigned number called the error tolerance.

The sufficient condition for convergent of the iteration process is stated below.

Theorem 2.2 *Let R be a region which contains the root (ξ, η). Let (x_0, y_0) be an initial guess to a root (ξ, η) of the equations $f(x, y) = 0, g(x, y) = 0$. If*

(i) the functions $f(x, y), g(x, y)$ and their first order partial derivatives are continuous and bounded in R, and

(ii) $J_n \neq 0$ in R,

then the sequence of approximation $x_{n+1} = x_n + h_n, y_{n+1} = y_n + k_n$, where h_n and k_n are given by (2.15), converges to the root (ξ, η).

Note 2.1 *The Newton-Raphson method reduces a pair of non-linear equations to a pair of linear equations in h and k. In general, this method converts a system of non-linear equations to a system of linear equations.*

Example 2.3 *Solve the pair of nonlinear equations $3x^2 - 2y^2 - 1 = 0$ and $x^2 - 2x + 2y - 8 = 0$ by Newton-Raphson method. The initial guess may be taken as $(2.5, 3.0)$.*

Solution. Let $f(x, y) = 3x^2 - 2y^2 - 1$ and $g(x, y) = x^2 - 2x + 2y - 8$.

$$\frac{\partial f}{\partial x} = 6x, \frac{\partial f}{\partial y} = 4y, \frac{\partial g}{\partial x} = 2x - 2, \frac{\partial g}{\partial y} = 2.$$

Therefore,

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 6x & -4y \\ 2x - 2 & 2 \end{bmatrix}.$$

First iteration

At (x_0, y_0),

$$\begin{bmatrix} -f_0 \\ -g_0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} 15 & -12 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} h_0 \\ k_0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

Since, $J_0 = \begin{vmatrix} 15 & -12 \\ 3 & 2 \end{vmatrix} = 66.$

By matrix inverse method,

$$\begin{bmatrix} h_0 \\ k_0 \end{bmatrix} = \frac{1}{66} \begin{bmatrix} 2 & 12 \\ -3 & 15 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0.143939 \\ 0.159091 \end{bmatrix}$$

Thus, $x_1 = x_0 + h_0 = 2.5 + 0.143939 = 2.643939$,
$y_1 = y_0 + k_0 = 3.0 + 0.159091 = 3.159091$. This is the first approximate root.

Second iteration

At (x_1, y_1),
$$\begin{bmatrix} -f_1 \\ -g_1 \end{bmatrix} = \begin{bmatrix} -0.011536 \\ -0.020719 \end{bmatrix}.$$

$$\begin{bmatrix} 15.863637 & -12.636364 \\ 3.287879 & 2.000000 \end{bmatrix} \begin{bmatrix} h_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} -f_1 \\ -g_1 \end{bmatrix}$$

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$J_1 = \begin{vmatrix} 15.863637 & -12.636364 \\ 3.287879 & 2.000000 \end{vmatrix} = 73.274109.$

Therefore,

$$\begin{bmatrix} h_1 \\ k_1 \end{bmatrix} = \frac{1}{73.274109} \begin{bmatrix} 2.000000 & 12.636364 \\ -3.287879 & 15.863637 \end{bmatrix} \begin{bmatrix} -0.011536 \\ -0.020719 \end{bmatrix} = \begin{bmatrix} -0.003888 \\ -0.003968 \end{bmatrix}.$$

Therefore, $x_2 = x_1 + h_1 = 2.643939 - 0.003888 = 2.640052$,
$y_2 = y_1 + k_1 = 3.159091 - 0.003968 = 3.155123$.

Third iteration

At (x_2, y_2),
$\begin{bmatrix} -f_2 \\ -g_2 \end{bmatrix} = \begin{bmatrix} -0.000015 \\ -0.000015 \end{bmatrix}.$

$$\begin{bmatrix} 15.840309 & -12.620492 \\ 3.280103 & 2.000000 \end{bmatrix} \begin{bmatrix} h_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} -f_2 \\ -g_2 \end{bmatrix}$$

Here, $J_2 = \begin{vmatrix} 15.840309 & -12.620492 \\ 3.280103 & 2.000000 \end{vmatrix} = 73.077133.$

Thus,

$$\begin{bmatrix} h_2 \\ k_2 \end{bmatrix} = \frac{1}{73.077133} \begin{bmatrix} 2.000000 & 12.620492 \\ -3.280103 & 15.840309 \end{bmatrix} \begin{bmatrix} -0.000015 \\ -0.000015 \end{bmatrix} = \begin{bmatrix} -0.000003 \\ -0.000003 \end{bmatrix}.$$

Hence, $x_3 = x_2 + h_2 = 2.640052 - 0.000003 = 2.640049$,
$y_3 = y_2 + k_2 = 3.155123 - 0.000003 = 3.155120$.

Thus, one root is $x = 2.64005, y = 3.155120$ correct up to five decimal places.

Note 2.2 *In iteration and Seidal methods, some pre-calculations are required to find the functions $\phi(x, y)$ and $\psi(x, y)$. But, finding of these functions is a very difficult task, particularly for three or more variables case.*

On the other hand, no precalculation is required for Newton-Raphson method. But, in Newton-Raphson method partial derivatives of the functions $f(x, y)$ and $g(x, y)$ are required. No derivatives are required in iteration and Seidal methods.

The rate of convergent of Newton-Raphson method is quadratic whereas iteration and Seidal methods it is linear.

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Self Assessment (MCQ/Short answer questions)

1. Let $x = \phi(x, y), y = \psi(x, y)$ be a pair of non-linear equations. Assume that their first order partial derivatives are continuous on the region R that contains a root (ξ, η). If the starting point (x_0, y_0) is sufficiently close to (ξ, η). Then the sufficient condition for convergence of the root, for all $(x, y) \in R$ is

   (a) $\left|\frac{\partial \phi}{\partial x}\right| + \left|\frac{\partial \phi}{\partial y}\right| < 1$ or $\left|\frac{\partial \psi}{\partial x}\right| + \left|\frac{\partial \psi}{\partial y}\right| < 1$

   (b) $|\phi'(x)| < 1$ and $|\psi'(x)| < 1$

   (c) $\left|\frac{\partial \phi}{\partial x}\right| + \left|\frac{\partial \phi}{\partial y}\right| < 1$ and $\left|\frac{\partial \psi}{\partial x}\right| + \left|\frac{\partial \psi}{\partial y}\right| < 1$

   (d) always convergent

2. The iteration scheme of the fixed point iteration method for the equations $x = \phi(x, y), y = \psi(x, y)$ is

   (a) $x_{n+1} = \phi(x_0, y_n), y_{n+1} = \psi(x_n, y_0)$

   (b) $x_{n+1} = \phi(x_n, y_n), y_{n+1} = \psi(x_{n+1}, y_n)$

   (c) $x_{n+1} = \phi(x_n, y_n), y_{n+1} = \psi(x_n, y_n)$

   (d) none of the above

3. The iteration scheme of the fixed point iteration method for the equations $x = \phi(x, y, z), y = \psi(x, y, z), z = \zeta(x, y, z)$ is

   (a) $x_{n+1} = \phi(x_0, y_n, z_n), y_{n+1} = \psi(x_n, y_0, z_n), z_{n+1} = \zeta(x_n, y_n, z_0)$

   (b) $x_{n+1} = \phi(x_n, y_n, z_n), y_{n+1} = \psi(x_n, y_n, z_n), z_{n+1} = \zeta(x_n, y_n, z_n)$

   (c) $x_{n+1} = \phi(x_n, y_n, z_n), y_{n+1} = \psi(x_{n+1}, y_n, z_n), z_{n+1} = \zeta(x_{n+1}, y_{n+1}, z_n)$

   (d) none of the above

4. The iteration scheme of the Seidal method for the equations $x = \phi(x, y, z), y = \psi(x, y, z), z = \zeta(x, y, z)$ is

   (a) $x_{n+1} = \phi(x_0, y_n, z_n), y_{n+1} = \psi(x_n, y_0, z_n), z_{n+1} = \zeta(x_n, y_n, z_0)$

   (b) $x_{n+1} = \phi(x_n, y_n, z_n), y_{n+1} = \psi(x_n, y_n, z_n), z_{n+1} = \zeta(x_n, y_n, z_n)$

   (c) $x_{n+1} = \phi(x_n, y_n, z_n), y_{n+1} = \psi(x_{n+1}, y_n, z_n), z_{n+1} = \zeta(x_{n+1}, y_{n+1}, z_n)$

   (b) none of the above

5. Let $x = \frac{8x-4x^2+y^2+1}{8}$ and $y = \frac{2x-x^2+4y-y^2+3}{4}$ be a pair of equations. If $(x_0, y_0) =$

$(1.1, 2.0)$ is an initial root, then the first approximate root (correct up to three decimal places) obtained by fixed point iteration method is

(a) $(1.120, 1.998)$ (b) $(1.5, 2)$ (c) $(1.232, 1.998)$ (d) $(1.120, 2.112)$

6. Let $x = \frac{8x-4x^2+y^2+1}{8}$ and $y = \frac{2x-x^2+4y-y^2+3}{4}$ be a pair of equations. If $(x_0, y_0) = (1.1, 2.0)$ is an initial root, then a root (correct up to four decimal places) obtained by Seidal method is

   (a) $(1.1200, 1.9964)$ (b) $(1.5234, 2.1100)$

   (c) $(1.2320, 1.9950)$ (d) $(1.1165, 1.9966)$

7. Newton-Raphson method is suitable if the initial guess (x_0, y_0) is closed to the exact root.

   (a) true (b) false

8. Seidal method is more faster than fixed point iteration method.

   (a) true (b) false

9. Is the rate of convergence of Newton-Rapshon method for a system of non-linear equations quadratic?

   (a) yes (b) no

10. Is $(1, 1.5)$ a root of the equations $x^2 - 2y + 2 = 0$ and $x^2 + y^2 + 3x - 2y = 0$?

    (a) yes (b) no

11. The sufficient condition for the convergence of the fixed point iteration method for the system of non-linear equation $x = \phi(x, y, z), y = \psi(x, y, z), z = \zeta(x, y, z)$ is $\cdots\cdots$.

12. The iteration scheme of Newton-Rapshon method for a pair of non-linear equations is $x_{n+1} = x_n + h, y_{n+1} = y_n + k$, where h and k are $\cdots\cdots$.

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Answer to the questions

1. (c)
2. (c)
3. (b)
4. (c)
5. (a)
6. (d)
7. (a)
8. (a)
9. (a)
10. (b)
11. $|\frac{\partial\phi}{\partial x}| + |\frac{\partial\phi}{\partial y}| + |\frac{\partial\phi}{\partial z}| < 1$, $|\frac{\partial\psi}{\partial x}| + |\frac{\partial\psi}{\partial y}| + |\frac{\partial\psi}{\partial z}| < 1$ and $|\frac{\partial\zeta}{\partial x}| + |\frac{\partial\zeta}{\partial y}| + |\frac{\partial\zeta}{\partial z}| < 1$.
12. $h = \frac{1}{J}\begin{vmatrix} -f_0 & \dfrac{\partial f_0}{\partial y} \\ -g_0 & \dfrac{\partial g_0}{\partial y} \end{vmatrix}$, $k = \frac{1}{J}\begin{vmatrix} \dfrac{\partial f_0}{\partial x} & -f_0 \\ \dfrac{\partial g_0}{\partial x} & -g_0 \end{vmatrix}$, where $J = \begin{vmatrix} \dfrac{\partial f_0}{\partial x} & \dfrac{\partial f_0}{\partial y} \\ \dfrac{\partial g_0}{\partial x} & \dfrac{\partial g_0}{\partial y} \end{vmatrix}$.

Self Assessment (Long answer questions)

1. Solve the following systems of nonlinear equations using iteration method
   (i) $x^2 + y = 11, y^2 + x = 7$
   (ii) $2xy - 3 = 0, x^2 - y - 2 = 0$.

2. Solve the following systems of nonlinear equations using Seidal method
   (i) $x^2 + 4y^2 - 4 = 0, x^2 - 2x - y + 1 = 0$, start with $(1.5, 0.5)$,
   (ii) $3x^2 - 2y^2 - 1 = 0, x^2 - 2x + y^2 + 2y - 8 = 0$, start with $(-1, 1)$.

3. Solve the following systems of nonlinear equations using Newton-Raphson method
   (i) $3x^2 - 2y^2 - 1 = 0, x^2 - 2x + 2y - 8 = 0$ start with initial guess $(2.5, 3)$,
   (ii) $x^2 - x + y^2 + z^2 - 5 = 0, x^2 + y^2 - y + z^2 - 4 = 0, x^2 + y^2 + z^2 + z - 6 = 0$ start with $(-0.8, 0.2, 1.8)$ and $(1.2, 2.2, -0.2)$.

4. Use Newtons method to find all nine solutions to
   $7x^3 - 10x - y - 1 = 0, 8y^3 - 11y + x - 1 = 0$.
   Use the starting points $(0, 0), (1, 0), (0, 1), (-1, 0), (0, -1), (1, 1), (-1, 1), (1, -1)$ and $(-1, -1)$.

Learn More

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