M.Sc. Course in

Applied Mathematics

with Oceanology and Computer Programming

Vidyasagar University

Semester-II

Paper-MTM 202 **Paper Name: Numerical Analysis**

Solution of System of Linear Equations

Module No. 2

Method of Matrix Factorization

Objective

(a) Factorization of matrix

(b) LU-decomposition method

(c) Limitation of LU-decomposition method

Keywords

matrix factorization, LU-decomposition

Developed by

Professor Madhumangal Pal

Department of Applied Mathematics

Vidyasagar University, Midnapore-721102

email: mmpalvu@gmail.com

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Let the system of linear equations be

$$\mathbf{Ax} = \mathbf{b} \tag{2.1}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{bmatrix}. \tag{2.2}$$

In the matrix factorization method, the coefficient matrix $\mathbf{A}$ is expressed as a product of two or more other matrices. By finding the factors of the coefficient matrix, some methods are adapted to solve a system of linear equations with less computational time. In this module, LU decomposition method is discussed to solve a system of linear equations. In this method, the coefficient matrix $\mathbf{A}$ is written as a product of two matrices $\mathbf{L}$ and $\mathbf{U}$, where the first matrix is a lower triangular matrix and second one is an upper triangular matrix.

2.1 LU decomposition method

LU decomposition method is also known as matrix factorization or Crout's reduction method.

Let the coefficient matrix $\mathbf{A}$ be written as $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}$ and $\mathbf{U}$ are the lower and upper triangular matrices respectively.

Unfortunately, this factorization is not possible for all matrices. Such factorization is possible and it is unique if all the principal minors of $\mathbf{A}$ are non-singular, i.e.

$$a_{11} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \quad \cdots, \quad |\mathbf{A}| \neq 0 \tag{2.3}$$

Since the matrices $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular, so these matrices can be written in the following form:

$$\mathbf{L} = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}. \tag{2.4}$$

If the factorization is possible, then the equation $\mathbf{Ax} = \mathbf{b}$ can be expressed as

$$\mathbf{LUx} = \mathbf{b}. \tag{2.5}$$

Let $\mathbf{Ux} = \mathbf{z}$, then the equation (2.5) reduces to $\mathbf{Lz} = \mathbf{b}$, where $\mathbf{z} = (z_1, z_2, \ldots, z_n)^t$ is an unknown vector. Thus, the equation (2.2) is decomposed into two systems of linear equations. Note that these systems are easy to solve.

The equation $\mathbf{Lz} = \mathbf{b}$ in explicit form is

$$\begin{array}{lcl} l_{11}z_1 & = & b_1 \\ l_{21}z_1 + l_{22}z_2 & = & b_2 \\ l_{31}z_1 + l_{32}z_2 + l_{33}z_3 & = & b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots \\ l_{n1}z_1 + l_{n2}z_2 + l_{n3}z_3 + \cdots + l_{nn}z_n & = & b_n. \end{array} \tag{2.6}$$

This system of equations can be solved by forward substitution, i.e. the value of z_1 is obtained from first equation and using this value, z_2 can be determined from second equation and so on. From last equation we can determine the value of z_n, as in this stage the values of the variables $z_1, z_2, \ldots, z_{n-1}$ are available.

By finding the values of $\mathbf{z}$, one can solve the equation $\mathbf{Ux} = \mathbf{z}$. In explicit form, this system is

$$\begin{array}{rcl} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 + \cdots + u_{1n}x_n & = & z_1 \\ u_{22}x_2 + u_{23}x_3 \cdots + z_{2n}x_n & = & z_2 \\ u_{33}x_3 + u_{23}x_3 \cdots + u_{3n}x_n & = & z_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots \\ u_{n-1n-1}x_{n-1} + u_{n-1n}x_n & = & z_{n-1} \\ u_{nn}x_n & = & z_n. \end{array} \tag{2.7}$$

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Observed that the value of the last variable x_n can be determined from the last equation. Using this value one can compute the value of x_{n-1} from the last but one equation, and so on. Lastly, from the first equation we can find the value of the variable x_1, as in this stage all other variables are already known. This process is called the backward substitution.

Thus, the outline to solve the system of equations $\mathbf{Ax} = \mathbf{b}$ is given. But, the complicated step is to determine the matrices $\mathbf{L}$ and $\mathbf{U}$. The matrices $\mathbf{L}$ and $\mathbf{U}$ are obtained from the relation $\mathbf{A} = \mathbf{LU}$. Note that, this matrix equation gives n^2 equations containing l_{ij} and u_{ij} for $i, j = 1, 2, \ldots, n$. But, the number of elements of the matrices $\mathbf{L}$ and $\mathbf{U}$ are $n(n+1)/2 + n(n+1)/2 = n^2 + n$. So, n additional equations/conditions are required to find $\mathbf{L}$ and $\mathbf{U}$ completely. Such conditions are discussed below.

When $u_{ii} = 1$, for $i = 1, 2, \ldots, n$, then the method is known as **Crout's decomposition method**. When $l_{ii} = 1$, for $i = 1, 2, \ldots, n$ then the method is known as **Doolittle's method** for decomposition. In particular, when $l_{ii} = u_{ii}$ for $i = 1, 2, \ldots, n$ then the corresponding method is called **Cholesky's decomposition method**.

2.1.1 Computation of $\mathbf{L}$ and $\mathbf{U}$

In this section, it is assumed that $u_{ii} = 1$ for $i = 1, 2, \ldots, n$. Now, the equation $\mathbf{LU} = \mathbf{A}$ becomes

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & \cdots & l_{11}u_{1n} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} & \cdots & l_{21}u_{1n} + l_{22}u_{2n} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} & \cdots & l_{31}u_{1n} + l_{32} + u_{2n} + l_{33}u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n1}u_{12} + l_{n2} & l_{n1}u_{13} + l_{n2}u_{23} + l_{n3} & \cdots & l_{n1}u_{1n} + l_{n2}u_{2n} + \cdots + l_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots & a_{nn} \end{bmatrix}.$$

From first row and first column, we have

$l_{i1} = a_{i1}$, $i = 1, 2, \ldots, n$ and $u_{1j} = \dfrac{a_{1j}}{l_{11}}$, $j = 2, 3, \ldots, n$.

Similarly, from second column and second row we get the following equations.

$$l_{i2} = a_{i2} - l_{i1}u_{12}, \quad \text{for } i = 2, 3, \ldots, n,$$
$$u_{2j} = \frac{a_{2j} - l_{21}u_{1j}}{l_{22}} \quad \text{for} \quad j = 3, 4, \ldots, n.$$

Solving these equations we obtained the second column of **L** and second row of **U**.

In general, the elements of the matrix **L**, i.e. l_{ij} and the elements of **U**, i.e. u_{ij} are determined from the following equations.

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}, \quad i \geq j \tag{2.8}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}}{l_{ii}}, \quad i < j \tag{2.9}$$

$$u_{ii} = 1, \quad l_{ij} = 0, \quad j > i \quad \text{and} \quad u_{ij} = 0, \quad i > j.$$

The matrix equations $\mathbf{Lz} = \mathbf{b}$ and $\mathbf{Ux} = \mathbf{z}$ can also be solved by finding the inverses of **L** and **U** as

$$\mathbf{z} = \mathbf{L^{-1}b} \tag{2.10}$$

$$\text{and} \quad \mathbf{x} = \mathbf{U^{-1}z}. \tag{2.11}$$

But, the process is time consuming, because finding of inverse takes much time.

It may be noted that the time to find the inverse of a triangular matrix is less than an arbitrary matrix.

The inverse of **A** can also be determined from the relation

$$\mathbf{A^{-1}} = \mathbf{U^{-1}L^{-1}}. \tag{2.12}$$

Few properties of triangular matrices

Let $\mathbf{L} = [l_{ij}]$ and $\mathbf{U} = [u_{ij}]$ be the lower and upper triangular matrices.

- The determinant of a triangular matrix is the product of the diagonal elements.
- Product of two lower (upper) triangular matrices is a lower (upper) triangular matrix.

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- Square of a lower (upper) triangular matrix is a lower (upper) triangular matrix.
- The inverse of lower (upper) triangular matrix is also a lower (upper) triangular matrix.
- Since $\mathbf{A} = \mathbf{LU}$, $|\mathbf{A}| = |\mathbf{L}||\mathbf{U}|$.

Let us illustrate the LU decomposition method.

Example 2.1 *Let*

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}.$$

Express $\mathbf{A}$ *as* $\mathbf{A} = \mathbf{LU}$*, where* $\mathbf{L}$ *and* $\mathbf{U}$ *are lower and upper triangular matrices and hence solve the system of equations* $2x_1 - 3x_2 + x_3 = 1$, $x_1 + 2x_2 - 3x_3 = 4$, $4x_1 - x_2 - 2x_3 = 8$.

Also, determine $\mathbf{L^{-1}}$, $\mathbf{U^{-1}}$, $\mathbf{A^{-1}}$ *and* $|\mathbf{A}|$.

Solution. Let

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}.$$

To find the values of l_{ij} and u_{ij}, comparing both sides and we obtained

$$\begin{array}{lll} l_{11} = 2, \ \ l_{21} = 1 & & l_{31} = 4 \\ l_{11}u_{12} = -3 & \text{or,} & u_{12} = -3/2 \\ l_{11}u_{13} = 1 & \text{or,} & u_{13} = 1/2 \\ l_{21}u_{12} + l_{22} = 2 & \text{or,} & l_{22} = 7/2 \\ l_{31}u_{12} + l_{32} = -1 & \text{or,} & l_{32} = 7 \\ l_{21}u_{13} + l_{22}u_{23} = -3 & \text{or,} & u_{23} = -1 \\ l_{31}u_{13} + l_{32}u_{23} + l_{33} = -2 & \text{or,} & l_{33} = 1. \end{array}$$

Hence $\mathbf{L}$ and $\mathbf{U}$ are given by

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 7/2 & 0 \\ 4 & 5 & 1 \end{bmatrix}, \qquad \mathbf{U} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The given equations can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}.$$

Let $\mathbf{A} = \mathbf{LU}$. Then, $\mathbf{LUx} = \mathbf{b}$. Let $\mathbf{Ux} = \mathbf{z}$. Then, the given equation reduces to $\mathbf{Lz} = \mathbf{b}$.

First we consider the equation $\mathbf{Lz} = \mathbf{b}$. Then

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 7/2 & 0 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}.$$

In explicit form these equations are

$$\begin{aligned} 2z_1 \qquad\qquad &= 1, \\ z_1 + 7/2z_2 \qquad &= 4, \\ 4z_1 + 5z_2 + z_3 &= 8. \end{aligned}$$

The solution of the above equations is $z_1 = 1/2, \;\; z_2 = 1, \;\; z_3 = 1$.

Therefore, $\mathbf{z} = (1/2, 1, 1)^t$.

Now, we solve the equation $\mathbf{Ux} = \mathbf{z}$, i.e.

$$\begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}.$$

In explicit form, the equations are

$$\begin{aligned} x_1 - (3/2)x_2 + (1/2)x_3 &= 1/2 \\ x_2 - x_3 &= 1 \\ x_3 &= 1. \end{aligned}$$

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The solution is $x_3 = 1$, $x_2 = 1 + 1 = 2$, $x_1 = 1/2 + (3/2)x_2 - (1/2)x_3 = 3$, i.e. $x_1 = 3, x_2 = 2, x_3 = 1$.

Third Part. Gauss-Jordan method is used to find $\mathbf{L^{-1}}$. Augmented matrix is

$$
\begin{aligned}
[\mathbf{L}\vdots\mathbf{I}] &= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 7/2 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & 7/2 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right] R_1' = \frac{1}{2}R_1 \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 7/2 & 0 & -1/2 & 1 & 0 \\ 0 & 5 & 1 & -2 & 0 & 1 \end{array}\right] R_2' = R_2 - R_1,\ R_3' = R_3 - 4R_1 \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/7 & 2/7 & 0 \\ 0 & 5 & 1 & -2 & 0 & 1 \end{array}\right] R_1' = \frac{2}{7}R_2 \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/7 & 2/7 & 0 \\ 0 & 0 & 1 & -9/7 & -10/7 & 1 \end{array}\right] R_3' = R_3 - 5R_2.
\end{aligned}
$$

Thus, $\mathbf{L^{-1}} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/7 & 2/7 & 0 \\ -9/7 & -10/7 & 1 \end{bmatrix}$.

Using same process, one can determine $\mathbf{U^{-1}}$. But, here another method is used to determine $\mathbf{U^{-1}}$. We know that the inverse of an upper triangular matrix is upper triangular.

Therefore, let $\mathbf{U^{-1}} = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$.

From the identity $\mathbf{U^{-1}U} = \mathbf{I}$, we have

$$\begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This gives,
$$\begin{bmatrix} 1 & -3/2 + b_{12} & 1/2 - b_{12} + b_{13} \\ 0 & 1 & -1 + b_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Comparing both sides

$-3/2 + b_{12} = 0$ or, $b_{12} = 3/2$, $1/2 - b_{12} + b_{13} = 0$ or, $b_{13} = 1$

$-1 + b_{23} = 0$ or, $b_{23} = 1$

Thus,

$$\mathbf{U^{-1}} = \begin{bmatrix} 1 & 3/2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now,

$$\begin{aligned} \mathbf{A^{-1}} &= \mathbf{U^{-1}L^{-1}} = \begin{bmatrix} 1 & 3/2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1/7 & 2/7 & 0 \\ -9/7 & -10/7 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & 1 \\ -10/7 & -8/7 & 1 \\ -9/7 & -10/7 & 1 \end{bmatrix}. \end{aligned}$$

Last Part. $|\mathbf{A}| = |\mathbf{L}||\mathbf{U}| = \Big(2 \times (7/2) \times 1\Big) \times 1 = 7.$

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Self Assessment (MCQ/Short answer questions)

1. A matrix A can be factorized as a product of a lower triangular matrix and an upper triangular matrix, if
   (a) A is non-singular
   (b) all principle minors are non-zero
   (c) A is only symmetric
   (d) none of these

2. Let A be a symmetric and positive definite matrix. U and L denote the upper and lower triangular matrices. Then A can be written as
   (a) $A = LL^t$ (b) $A = U^2$ (c) $A = LU$ (d) $A = L^2$

3. LU-decomposition method is a direct method.
   (a) true (b) false

4. The inverse of an upper triangular matrix is upper triangular.
   (a) true (b) false

5. Is LU-decomposition method applicable to all types of system of linear equations?
   (a) yes (b) no

6. If $A = LU = \begin{bmatrix} 2 & -2 & 1 \\ 5 & 1 & -3 \\ 3 & 4 & 1 \end{bmatrix}$. Then the lower triangular matrix L is $\cdots\cdots$.

Answer to the questions

1. (b)
2. (a)
3. (a)
4. (a)
5. (b)
6. $\begin{bmatrix} 1.41421 & 0 & 0 \\ 0.70711 & 1.58114 & 0 \\ 0.70711 & 0.94868 & 1.61245 \end{bmatrix}$

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Self Assessment (Long answer questions)

1. Using LU decomposition method, solve the following systems of equations

   (i) $x_1 + x_2 + x_3 = 3$

   $2x_1 - x_2 + 3x_3 = 16$

   $3x_1 + x_2 - x_3 = -3$

   (ii) $x2y + 7z = 6$

   $4x + 2y + z = 7$

   $2x + 5y2z = 5.$

2. Find the triangular factorization $A = LU$ for the matrices

   (i) $\begin{pmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 7 \end{pmatrix}$

   (ii) $\begin{pmatrix} 5 & 2 & 1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{pmatrix}$

3. Solve $LY = B$, $UX = Y$ and verify that $B = AX$ for $B = (8, -4, 10, -4)^t$, where $A = LU$ is given by

$$\begin{pmatrix} 4 & 8 & 4 & 0 \\ 1 & 5 & 4 & -3 \\ 1 & 4 & 7 & 2 \\ 1 & 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & 2/3 & 1 & 0 \\ 1/4 & 1/3 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & 4 & 0 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Learn More

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