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**Applied Mathematics**

**with Oceanology and Computer Programming**

**Vidyasagar University**

Semester-II

**Paper-MTM 202 Paper Name: Numerical Analysis**

# Solution of System of Linear Equations

## Module No. 1

# Solution of System of Linear Equations by Matrix Inverse Method

***Objective***

(a) Partial pivoting, complete pivoting

(b) Evaluation of determinant by partial pivoting

(c) Evaluation of inverse of matrix by partial pivoting

(d) Solution of system of equations by matrix inverse method

***Keywords***

partial pivoting, complete pivoting, Gauss-Jordon method, matrix inverse

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The system of linear and non-linear equations occur in many applications. To solve a system of linear equations many direct and iterated methods are developed. The old and trivial methods are Cramer's rule and matrix inverse method. But, these methods depend on evaluation of determinant and computation of inverse of the coefficient matrix. Few methods are available to evaluate a determinant, among them pivoting method is most efficient and applicable for all type of determinants. In this module, pivoting method is discussed to evaluate a determinant and inverse of the coefficient matrix. Then, matrix inverse method is described to solve a system of linear equations. Other direct and iteration methods are discussed in next modules.

A system of $m$ linear equations with $n$ variables is given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \quad \cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{1.1}$$

The quantities $x_1$, $x_2$, …, $x_n$ are the **unknowns** (**variables**) of the system and $a_{11}$, $a_{12}$, …, $a_{mn}$ are called the **coefficients** and generally they are known. The numbers $b_1, b_2, \ldots, b_m$ are **constant** or **free terms** of the system.

The above system of equations (1.1) can be written as a single equation:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, 2, \ldots, m. \tag{1.2}$$

Also, the entire system of equations (1.1) can be written with the help of matrices as

$$\mathbf{AX} = \mathbf{b}, \tag{1.3}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{bmatrix}. \tag{1.4}$$

A system of linear equations may or may not have a solution. If the system of linear equations (1.1) has a solution then the system is called **consistent** otherwise it is called **inconsistent** or **incompatible**. Again, a consistent system of linear equations may have unique solution or multiple solutions. Finding of unique solution is easy, but determination of multiple solutions, if exists, is a complicated problem.

To solve a system of linear equations usually three type of the elementary transformations are applied. These are discussed below.

**Interchange:** The order of two equations can be changed.

**Scaling:** Multiplication of both sides of an equation by any non-zero number.

**Replacement:** Addition to (subtraction from) both sides of one equation of the corresponding sides of another equation multiplied by any number.

If for a system, all the constant terms $b_1, b_2, \ldots, b_m$ are zero, then the system is called homogeneous system otherwise it is called the non-homogeneous system.

Two type of methods are available to solve a system of linear equations, viz. direct method and iteration method.

Again, many direct methods are used to solve a system of equations, among them Cramer's rule, matrix inversion, Gauss elimination, matrix factorization, etc. are well known.

Also, the mostly used iteration methods are Jacobi's iteration, Gauss-Seidal's iteration, etc.

In many applications, we have to determine the value of a determinant. So an efficient method is required for this purpose. One efficient method based on pivoting is discussed in the following section.

## 1.1 Evaluation of determinant

One of the best methods to evaluate determinant is known as triangularization and it is also known as Gauss reduction method. The main idea of this method is to convert the given determinant ($D$) into a lower or upper triangular form by using only elementary row operations. If the determinant is reduced to a triangular form (say $D'$), then the value of $D$ is obtained by multiplying the diagonal elements of $D'$.

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Let $D$ be a determinant of order $n$ given by $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Using the elementary row operations, $D$ can be reduced to the following upper triangular form:

$$D' = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} \end{vmatrix}.$$

To convert in this form lot of elementary operations are required. To convert all the elements of the first column, except first element, to 0 the following elementary operations are used

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}, \text{ for } i, j = 2, 3, \ldots, n.$$

Similarly, to convert all the elements of the second column below the second element to 0, the following operations are used.

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} a_{2j}^{(1)}, \text{ for } i, j = 3, 4, \ldots, n.$$

All these elementary operations can be written as

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}; \tag{1.5}$$

$i, j = k+1, \ldots, n$; $k = 1, 2, \ldots, n-1$ and $a_{ij}^{(0)} = a_{ij}$, $i, j = 1, 2, \cdots, n$.

Once $D'$ is available, then the value of $D$ is given by

$$a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{nn}^{(n-1)}.$$

It is observed that the formula for the elementary operations is simple and easy to programmed. The time taken by this method is $O(n^3)$. But, there is a serious drawback of this formula, which is discussed below.

To compute the value of $a_{ij}^{(k)}$ one division is required. If $a_{kk}^{(k-1)}$ is zero or very small then the method fails. If $a_{kk}^{(k-1)}$ is very small, then there is a chance of loosing significant digits or data overflow. To avoid this situation the pivoting techniques are used.

A **pivot** is the largest magnitude element in a row or in a column or in the principal diagonal or the leading or trailing sub-matrix of order $i$ $(2 \leq i \leq n)$.

Let us consider the following matrix to illustrate these terms:

$$A = \begin{bmatrix} 0 & 1 & 0 & -5 \\ 1 & -8 & 3 & 10 \\ 9 & 3 & -33 & 18 \\ 4 & -40 & 9 & 11 \end{bmatrix}.$$

For this matrix 9 is the pivot for the first column, $-33$ is the pivot for the principal diagonal, $-40$ is the pivot for the entire matrix and $-8$ is the pivot for the trailing sub-matrix $\begin{bmatrix} 0 & 1 \\ 1 & -8 \end{bmatrix}$.

If any one of the column pivot element (during elementary operation) is zero or very small relative to other elements in that row, then we rearrange the remaining rows in such a way that the pivot becomes non-zero or not a very small number. The method is called pivoting. The pivoting methods are of two types, viz. partial pivoting and complete pivoting, these are discussed below.

### 1.1.1 Partial pivoting

In partial pivoting method, the pivot is the largest magnitude element in a column. In the first stage, find the first pivot which is the largest element in magnitude among the elements of first column. If it is $a_{11}$, then there is nothing to do. If it is $a_{i1}$, then interchange rows $i$ and 1. Then apply the elementary row operations to make all the elements of first column, except first element, to 0. In the next stage, the second pivot is determined by finding the largest element in magnitude among the elements of second column leaving first element and let it be $a_{j2}$. In this case, interchange second and $j$th rows and then apply elementary row operations. This process continues for $(n-1)$th times. In general, at the $i$th stage, the smallest index $j$ is chosen for which

$$|a_{ij}^{(k)}| = \max\{|a_{kk}^{(k)}|, |a_{k+1\,k}^{(k)}|, \ldots, |a_{nk}^{(k)}|\} = \max\{|a_{ik}^{(k)}|, i = k, k+1, \ldots, n\}$$

and the rows $k$ and $j$ are interchanged.

**Complete pivoting or full pivoting**

In partial pivoting, the pivot is chosen from column. But, in complete pivoting the pivot element is the largest element (in magnitude) among all the elements of the determinant. Let it be at the $(l, m)$th position for first time.

Thus, $a_{lm}$ is the first pivot. Then interchange first row and the $l$th row and of first column and $m$th column. In second stage, the largest element (in magnitude) is determined among all elements leaving the first row and first column. This element is the second pivot.

In this manner, at the $k$th stage, we choose $l$ and $m$ such that

$$|a_{lm}^{(k)}| = \max\{|a_{ij}^{(k)}|, i, j = k, k+1, \ldots, n\}.$$

Then interchange the rows $k$, $l$ and columns $k$, $m$. In this case, $a_{kk}$ is the $k$th pivot element.

It is obvious that the complete pivoting is more complicated than the partial pivoting. Partial pivoting is easy to program. Generally, partial pivoting is used for hand calculation.

We have mentioned earlier that the pivoting is used to find the value of all kind of determinants. To determine the pivot and to interchange the rows and/or columns some additional time is required. But, for some type of determinants without pivoting one can determine its value. Such type of determinants are stated below.

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**Note 1.1** *If the coefficient matrix* $\mathbf{A}$ *is diagonally dominant, i.e.*

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}| \qquad \text{or} \qquad \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}| < |a_{ii}|, \qquad \text{for } i = 1, 2, \ldots, n. \tag{1.6}$$

*or real symmetric and positive definite then no pivoting is necessary.*

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**Note 1.2** *Every diagonally dominant matrix is non-singular.*

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**Example 1.1** *Convert the determinant*

$$\mathbf{A} = \begin{vmatrix} 1 & 0 & 3 \\ -2 & 7 & 1 \\ 5 & -1 & 6 \end{vmatrix}$$

*into the upper triangular form using (i) partial pivoting, and (ii) complete pivoting and hence determine the value of* $\mathbf{A}$*.*

**Solution.** (i) (Partial pivoting) The largest element in the first column is 5, present in the third row and it is the first pivot of $\mathbf{A}$. Therefore, first and third rows are interchanged and the reduced determinant is

$$\begin{vmatrix} 5 & -1 & 6 \\ -2 & 7 & 1 \\ 1 & 0 & 3 \end{vmatrix}.$$

Since two rows are interchanged then the value of the determinant is to be multiplied by $-1$. To maintain it a variable *sign* is used and in this case it's value is $sign = -1$.

Now, we apply the elementary row operations to convert all elements of first column, except first, to 0.

Adding $\frac{2}{5}$ times the first row to the second row, $-\dfrac{1}{5}$ times the first row to the third row, i.e. $R_2' = R_2 + \dfrac{2}{5}R_1$ and $R_3' = R_3 - \dfrac{1}{5}R_1$. ($R_2$ and $R_2'$ represent the original second row and modified second row respectively.)

The reduced determinant is

$$\begin{vmatrix} 5 & -1 & 6 \\ 0 & 33/5 & 17/5 \\ 0 & 1/5 & 9/5 \end{vmatrix}.$$

Now, we determine the second pivot element. In this case, the pivot element is at the $(2,2)$th position, therefore no interchange is required.

Adding $\dfrac{-1/5}{33/5} = -\dfrac{1}{33}$ times the second row to the third row, i.e. $R_3' = R_3 - \dfrac{1}{33}R_2$. The reduced determinant is

$$\begin{vmatrix} 5 & -1 & 6 \\ 0 & 33/5 & 17/5 \\ 0 & 0 & 56/33 \end{vmatrix}.$$

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Note that this is an upper triangular determinant and hence its value is $sign \times (5)(33/5)(56/33) = -56$.

(ii) (Complete pivoting) The largest element in $\mathbf{A}$ is 7 at position (2,2). Interchanging first and second columns and assign $sign = -1$; and then interchanging first and second rows and setting $sign = -sign = 1$. Then the updated determinant is

$$\begin{vmatrix} 7 & -2 & 1 \\ 0 & 1 & 3 \\ -1 & 5 & 6 \end{vmatrix}.$$

Adding $\frac{1}{7}$ times the first row to the third row, i.e. using the formula $R_3' = R_3 + \frac{1}{7}R_1$. The reduced determinant is

$$\begin{vmatrix} 7 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 33/7 & 43/7 \end{vmatrix}.$$

Now, we determine the second pivot element from the submatrix obtained by deleting first row and column. That is, from the trailing sub-matrix $\begin{vmatrix} 1 & 3 \\ 33/7 & 43/7 \end{vmatrix}$.

The second pivot is 43/7 at (3,3) position. Interchange the second and third columns and setting $sign = -sign = -1$ and then interchanging second and third rows. Then the modified determinant is $\begin{vmatrix} 7 & 1 & -2 \\ 0 & 43/7 & 33/7 \\ 0 & 3 & 1 \end{vmatrix}$ and $sign = 1$.

Now, we apply row operation as $R_3' = R_3 - \dfrac{21}{43}R_2$ and we obtain the required upper triangular determinant $\begin{vmatrix} 7 & 1 & -2 \\ 0 & 43/7 & 33/7 \\ 0 & 0 & -56/43 \end{vmatrix}$.

Hence, the value of the determinant is $sign \times (7)(43/7)(-56/43) = -56$.

Observed that the values obtained by both the methods are same and it is expected.

### Advantages and disadvantages of partial and complete pivoting

In pivoting method, the symmetry or regularity of the original matrix may be lost. It is easily observed that the partial pivoting requires less time, as it needs less number of interchanges than complete pivoting. Again, the partial pivoting needs less number of comparison to get pivot element. A combination of partial and complete pivoting is expected to be very effective not only for computing a determinant but also for solving system of linear equations. The pivoting prevent the loss of significant digits.

## 1.2 Inverse of a matrix

Let $\mathbf{A}$ be a non-singular square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$. Then $\mathbf{B}$ is called the inverse of $\mathbf{A}$ and vice-versa. The inverse of a matrix is denoted by $\mathbf{A^{-1}}$. Now, using some theories of matrices it can be shown that the inverse of a matrix $\mathbf{A}$ is given by

$$\mathbf{A^{-1}} = \frac{\textbf{adj } \mathbf{A}}{|\mathbf{A}|}. \tag{1.7}$$

The matrix **adj A** is called adjoint of $\mathbf{A}$ and defined as

$$\textbf{adj } \mathbf{A} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix},$$

where $A_{ij}$ being the cofactor of $a_{ij}$ in $|\mathbf{A}|$.

This is the first definition to find the inverse of a matrix.

But, this definition is not suitable for large matrix as it needs huge amount of arithmetic calculations. In this method, we have to calculate $n^2$ cofactors and each cofactor is a determinant of order $(n-1) \times (n-1)$. It is mentioned in previous section that to evaluate a determinant of order $n$, $O(n^3)$ arithmetic calculations are required. Thus, to compute all cofactors, total $(n^3 \times n^2) = O(n^5)$ arithmetic calculations are needed. This is a huge amount of time for large matrices.

Fortunately, many efficient methods are available to find the inverse of a matrix, among them **Gauss-Jordan** is most popular. In the following Gauss-Jordan method is discussed to find the inverse of a square non-singular matrix.

### 1.2.1 Gauss-Jordan method

In this method, the matrix $\mathbf{A}$ is augmented with a unit matrix of same size, and only elementary row operations are applied to get the inverse of the matrix. Let the order of the matrix $\mathbf{A}$ be $n \times n$ and it is augmented with the unit matrix $\mathbf{I}$. This augmented matrix is denoted by $[\mathbf{A}\vdots\mathbf{I}]$. The order of the augmented matrix $[\mathbf{A}\vdots\mathbf{I}]$ becomes $n \times 2n$. The augmented matrix is of the following form:

$$[\mathbf{A}\vdots\mathbf{I}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \vdots & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{1.8}$$

Now, the inverse of $\mathbf{A}$ is calculated in two phases. In the first phase, the first half of the augmented matrix is converted into an upper triangular matrix by using only elementary row operations. In the second phase, this upper triangular matrix is converted to an identity matrix by using only row operations. All these operations are applied on the augmented matrix $[\mathbf{A}\vdots\mathbf{I}]$.

After second phase, the augmented matrix $[\mathbf{A}\vdots\mathbf{I}]$ is transferred to $[\mathbf{I}\vdots\mathbf{A^{-1}}]$. Thus, the right half becomes the inverse of $\mathbf{A}$. Symbolically, we can write as

$$[\mathbf{A}\vdots\mathbf{I}] \xrightarrow{\text{Gauss} - \text{Jordan}} [\mathbf{I}\vdots\mathbf{A^{-1}}].$$

In explicit form, the transformation is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \vdots & 0 & 0 & \cdots & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & \cdots & 0 & \vdots & a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & 1 & \cdots & 0 & \vdots & a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \vdots & a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{bmatrix}.$$

**Example 1.2** *Use partial pivoting method to find the inverse of the following matrix*
$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 4 & -2 & 0 \end{bmatrix}$.

**Solution.** The augmented matrix $[\mathbf{A} \vdots \mathbf{I}]$ is

$$[\mathbf{A} \vdots \mathbf{I}] = \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 4 & -2 & 0 & 0 & 0 & 1 \end{array}\right].$$

**Phase 1.** *(Reduction to upper triangular form):*

In the first column 4 is the largest element, so it is the first pivot. So we interchange first and third rows to place the pivot element 4 at the (1,1) position. Then, the above matrix becomes

$$\left[\begin{array}{ccc|ccc} 4 & -2 & 0 & 0 & 0 & 1 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 0 & 0 & 1/4 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right] R_1' = \frac{1}{4}R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 0 & 0 & 1/4 \\ 0 & 5/2 & 4 & 0 & 1 & 1/4 \\ 0 & 1 & 1 & 1 & 0 & -1/2 \end{array}\right] R_2' = R_2 + R_1; \quad R_3' = R_3 - 2R_1$$

All the elements of first column, except first, become 0. Now, we convert the element of (3,2) position to 0. For this purpose, we find the largest element (in magnitude) from the second column leaving first element and it is $\frac{5}{2}$. Fortunately, it is at (2,2) position and so there is no need to interchange any rows.

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 8/5 & 0 & 2/5 & 1/10 \\ 0 & 1 & 1 & 1 & 0 & -1/2 \end{array}\right] R_2' = \frac{2}{5}R_2$$

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$$\sim \left[\begin{array}{ccc:ccc} 1 & -1/2 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 8/5 & 0 & 2/5 & 1/10 \\ 0 & 0 & -3/5 & 1 & -2/5 & -3/5 \end{array}\right] R_3' = R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & -1/2 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 8/5 & 0 & 2/5 & 1/10 \\ 0 & 0 & 1 & -5/3 & 2/3 & 1 \end{array}\right] R_3' = -\frac{5}{3}R_2$$

**Phase 2.** *(Make the left half a unit matrix):*

$$[A\vdots I] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 4/5 & 0 & 1/5 & 3/10 \\ 0 & 1 & 8/5 & 0 & 2/5 & 1/10 \\ 0 & 0 & 1 & -5/3 & 2/3 & 1 \end{array}\right] R_1' = R_1 + \frac{1}{2}R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 4/3 & -1/3 & -1/2 \\ 0 & 1 & 0 & 8/3 & -2/3 & -3/2 \\ 0 & 0 & 1 & -5/3 & 2/3 & 1 \end{array}\right] R_1' = R_1 - \frac{4}{5}R_3 \ ; \quad R_2' = R_2 - \frac{8}{5}R_3$$

Now, the left half becomes a unit matrix, thus the second half is the inverse of the given matrix, and it is

$$\begin{bmatrix} 4/3 & -1/3 & -1/2 \\ 8/3 & -2/3 & -3/2 \\ -5/3 & 2/3 & 1 \end{bmatrix}.$$

### Complexity of the algorithm

By analyzing each step of the method to find the inverse of a matrix $\mathbf{A}$ of order $n \times n$, it can be shown that the time complexity to compute the inverse of a non-singular matrix is $O(n^3)$.

## 1.3 Matrix inverse method

A system of equations (1.1) can be written in the matrix form (1.3) as

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}, \mathbf{b}$ and $\mathbf{x}$ are defined in (1.4).

The solution of $\mathbf{Ax} = \mathbf{b}$ is obtained from the equation

$$\mathbf{x} = \mathbf{A^{-1}b} \tag{1.9}$$

where $\mathbf{A^{-1}}$ is the inverse of the matrix $\mathbf{A}$.

Thus, the vector $\mathbf{x}$ can be obtained by finding inverse of $\mathbf{A}$ and then multiplying with $\mathbf{b}$.

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**Example 1.3** *Solve the following system of equations by matrix inverse method*
$x_1 + 12x_2 + 3x_3 - 4x_4 + 6x_5 = 2,$
$13x_1 + 4x_2 + 5x_3 + 4x_5 = 4,$
$5x_1 + 4x_2 + 3x_3 + 2x_4 - 2x_5 = 6,$
$5x_1 + 14x_2 + 3x_4 - 2x_5 = 10,$
$-5x_1 + 4x_2 + 3x_3 + 4x_4 + 5x_5 = 13.$

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**Solution.** The given equations can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 12 & 3 & -4 & 6 \\ 13 & 4 & 5 & 0 & 4 \\ 5 & 4 & 3 & 2 & -2 \\ 5 & 14 & 0 & 3 & -2 \\ -5 & 4 & 3 & 4 & 5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 10 \\ 13 \end{bmatrix}.$$

Using partial pivoting method, the inverse of $\mathbf{A}$ is obtained as

$$\mathbf{A^{-1}} = \begin{bmatrix} -0.0362 & 0.0788 & -0.0641 & 0.0357 & -0.0309 \\ 0.0358 & -0.0241 & 0.0068 & 0.0464 & -0.0024 \\ 0.0798 & -0.0646 & 0.3333 & -0.1531 & 0.0280 \\ -0.1186 & 0.0473 & -0.0682 & 0.0768 & 0.1079 \\ -0.0178 & 0.0990 & -0.2150 & 0.0291 & 0.0679 \end{bmatrix}$$

Thus, the solution vector is $\mathbf{x} = \mathbf{A^{-1}b} = \begin{bmatrix} -0.1872 \\ 0.4486 \\ 0.7333 \\ 1.7136 \\ 0.2430 \end{bmatrix}$

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Hence, $x_1 = -0.1872, x_2 = 0.4486, x_3 = 0.7333, x_4 = 1.7136, x_5 = 0.2430$, correct up to four decimal places.

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**Note 1.3** *It is mentioned earlier that the time to compute the inverse of an $n \times n$ matrix is $O(n^3)$ and this amount of time is required to multiply two matrices of same order. Hence, the time complexity to solve a system of linear equations containing $n$ equations is $O(n^3)$.*

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## Self Assessment (MCQ/Short answer questions)

1. The first partial pivot of the matrix $\begin{bmatrix} 1 & 1 & 4 \\ 9 & 13 & 6 \\ -10 & 3 & 4 \end{bmatrix}$ is

   (a) 1 (b) 9 (c) $-10$ (d) 13

2. The first complete pivot of the matrix $\begin{bmatrix} 1 & 1 & -24 \\ 9 & 4 & 16 \\ 10 & 3 & 4 \end{bmatrix}$ is

   (a) 1 (b) 24 (c) 20 (d) $-24$

3. If a pivot is 0 at any stage during the evaluation of a determinant, then the value of the determinant is

   (a) $\infty$ (b) 0 (c) 1 (d) no conclusion can be drawn

4. Time complexity to find the inverse of a non-singular matrix of order $n \times n$ is

   (a) $O(n^3)$ (b) $O(n^2)$ (c) $O(n)$ (d) none of these

5. To find the inverse of a non-singular matrix using Gauss-Jordan method needs less arithmetic calculation than conventional method (i.e. using the formula $A^{-1} = adj\ A/|A|$),

   (a) true (b) false

6. If a matrix is diagonally dominant, then no pivoting is required.

   (a) true (b) false

7. If a matrix is positive definite, then no pivoting is required.

   (a) true (b) false

8. Without partial or complete pivoting method the value of a determinant cannot be determine.

   (a) true (b) false

9. Is complete pivoting method easier than partial pivoting method?

   (a) yes (b) no

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

10. Is pivoting required to find the value of a real symmetric determinant?
    (a) yes (b) no

11. If the first partial pivot element $a_{11}$ of a square matrix $A = [a_{ij}]$ is zero, then the value of the determinant is $\cdots\cdots$.

12. Let $A$ be a non-singular matrix of order $n \times n$. Now, the augmented matrix $[A : I]$ is reduced to $[I : B]$, $I$ is the unit matrix, by Gauss-Jordan method, then the inverse of $A$ is $\cdots\cdots$.

13. The value of the determinant

$$\begin{vmatrix} 10 & 1 & -4 & 2 \\ 0 & 24 & 6 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 5 \end{vmatrix} \quad \text{is } \cdots\cdots.$$

## Answer to the questions

1. (c)
2. (d)
3. (b)
4. (a)
5. (a)
6. (a)
7. (a)
8. (b)
9. (b)
10. (b)
11. 0
12. $B$
13. $-2640$

## Self Assessment (Long answer questions)

1. Find the inverse of the matrix using partial pivoting

$$\begin{pmatrix} 11 & 3 & 1 \\ 2 & 5 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

and hence solve the following system of equations. $11x_1 + 3x_2x_3 = 15$
$2x_1 + 5x_2 + 5x_3 = 11$ $x_1 + x_2 + x_3 = 1$.

2. Find the inverses of the following matrices (using partial pivoting).

(i) $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 5 & 1 \\ 6 & 8 & 9 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 5 \\ 3 & 8 & 7 \end{pmatrix}$

## Learn More


1. Danilina, N.I., Dubrovskaya, S.N., and Kvasha, O.P., and Smirnov, G.L. Computational Mathematics. Moscow: Mir Publishers, 1998.

2. Hildebrand, F.B. Introduction of Numerical Analysis. New York: London: McGraw-Hill, 1956.

3. Householder, A.S. The Theory of Matrices in Numerical Analysis. New York: Blaisdell, 1964.

4. Jain, M.K., Iyengar, S.R.K., and Jain, R.K. Numerical Methods for Scientific and Engineering Computation. New Delhi: New Age International (P) Limited, 1984.

5. Krishnamurthy, E.V., and Sen, S.K. Numerical Algorithms. New Delhi: Affiliated East-West Press Pvt. Ltd., 1986.

6. Mathews, J.H. Numerical Methods for Mathematics, Science, and Engineering, 2nd ed., NJ: Prentice-Hall, Inc., 1992.

7. Pal, M. Numerical Analysis for Scientists and Engineers: Theory and C Programs. New Delhi: Narosa, Oxford:Alpha Sciences, 2007.