M.Sc. Course in
Applied Mathematics
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Vidyasagar University

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Paper-MTM 202 **Paper Name: Numerical Analysis**

Numerical Solution of Partial Differential Equations

Module No. 3

Partial Differential Equation: Elliptic

Objective

(a) Finite difference method for Laplace equation

(b) Finite difference method for Poisson equation

(c) Five-point formulae

(d) Successive over-relaxation method

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Elliptic PDE is one of the widely used PDEs. In this module, the finite difference method is described to solve the elliptic PDEs.

3.1 Elliptic equations

The elliptic PDEs occur in many practical situations. The most simple elliptic PDEs are Laplace and Poisson equations. The Laplace equation is $\nabla^n u = 0$ and Poisson equation is $\nabla^n u = g(\mathbf{r})$.

One physical example of such equations is stated below.

Let the function ρ represent the electric charge density in some open bounded set $\Omega \subset R^d$. If the permittivity ε is constant in Ω the distribution of the electric potential φ in Ω is governed by the Poisson equation

$$-\varepsilon \Delta \varphi = \rho.$$

This equation does not have a unique solution, because if ϕ is a solution of this equation, then the function $\phi + c$, is also a solution, where c is any constant. To get a solution, every elliptic equation should have a suitable boundary conditions.

Let us consider the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ within the region } R \tag{3.1}$$
$$\text{and} \quad u = f(x, y) \text{ on the boundary } C.$$

In this problem, both are space variables. Now, we approximate this PDE by the central difference approximation. Then the finite difference approximation of the above equation is

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0. \tag{3.2}$$

It is assumed that the length of the subintervals along x and y directions are equal, i.e. $h = k$. Then the above equation becomes,

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}]. \tag{3.3}$$

From this expression, it is seen that the value of $u_{i,j}$ is the average of the values of u at the four meshes – north $(i, j+1)$, east $(i+1, j)$, south $(i, j-1)$ and west $(i-1, j)$.

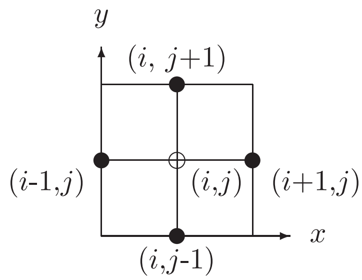


Figure 3.1: Known and unknown meshes in standard five-point formula

The known values (filled circles) and unknown (circle) are shown in Figure 3.1. This formula is known as **standard five-point formula**.

It can be proved that the Laplace equation remains invariant when the coordinates axes are rotated at an angle 45^0.

Under this rotation, the equation (3.3) becomes

$$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}]. \tag{3.4}$$

This is another formula to calculate $u_{i,j}$ and it is known as **diagonal five-point formula**. The known and unknown meshes for this formula are shown in Figure 3.2.

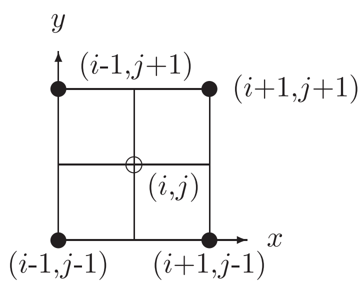


Figure 3.2: Known and unknown meshes in diagonal five-point formula

When the right hand side of the Laplace equation is nonzero, then this equation is known as Poisson's equation. The Poisson's equation in two-dimension is in the following form

$$u_{xx} + u_{yy} = g(x, y), \tag{3.5}$$

with the boundary condition $u = f(x, y)$ along the boundary C.

Here, we also assumed that the mesh points in both x and y directions are uniform. Using this assumption the central difference approximation of the equation (3.5) is reduced to

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 g_{i,j}] \text{ where } g_{i,j} = g(x_i, y_j). \qquad (3.6)$$

Let $u = 0$ along the boundary C and $i, j = 0, 1, 2, 3, 4$. Then $u_{0,j} = 0, u_{4,j} = 0$ for $j = 0, 1, 2, 3, 4$ and $u_{i,0} = 0, u_{i,4} = 0$ for $i = 0, 1, 2, 3, 4$. The boundary values (filled circles) are shown in Figure 3.3.

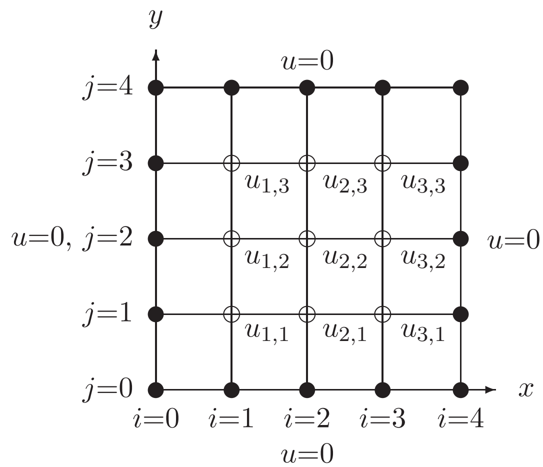


Figure 3.3: The 5×5 meshes for elliptic equation

For a particular case, i.e. for $i, j = 1, 2, 3$ the equation (3.6) becomes a system of nine equations with nine unknowns. These equations are written in matrix notation as

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} -h^2 g_{1,1} \\ -h^2 g_{1,2} \\ -h^2 g_{1,3} \\ -h^2 g_{2,1} \\ -h^2 g_{2,2} \\ -h^2 g_{2,3} \\ -h^2 g_{3,1} \\ -h^2 g_{3,2} \\ -h^2 g_{3,3} \end{bmatrix} \qquad (3.7)$$

This indicates that the equation (3.6) is a system of N (where N is the number of subintervals along x and y directions) equations. Note that the coefficient matrix is symmetric, positive definite and sparse (many elements are 0). Since, the coefficient matrix is sparse, so it is suggested to use iterative method rather than direct method to solve the above system of equations. Commonly used iterative methods are Jacobi's method, Gauss-Seidel's method, successive overrelaxation method, alternate direction implicit method, etc.

3.1.1 Method to find first approximate value of Laplace's equation

Let us consider the Laplace's equation $u_{xx} + u_{yy} = 0$. Let the region R be square and it is divided into $N \times N$ small squares each of side h. The boundary values are $u_{0,j}, u_{N,j}, u_{i,0}, u_{i,N}$ where $i, j = 0, 1, 2, \ldots, N$, shown in Figure 3.4.

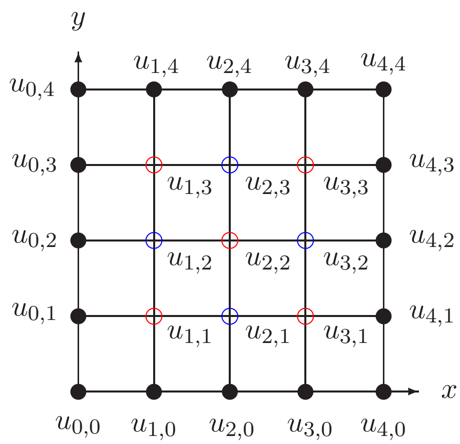


Figure 3.4: Known and unknown meshes for Laplace equation. Red meshes are calculated by diagonal five-point formula and blue meshes are determined by standard five-point formula

At first the diagonal five-point formula is used to compute the values of u according to the order $u_{2,2}, u_{1,3}$, $u_{3,3}$, $u_{1,1}$ and $u_{3,1}$ (red meshes in the figure). That is,

$$u_{2,2} = \frac{1}{4}(u_{0,0} + u_{4,4} + u_{0,4} + u_{4,0})$$
$$u_{1,3} = \frac{1}{4}(u_{0,2} + u_{2,4} + u_{0,4} + u_{2,2})$$

$$u_{3,3} = \frac{1}{4}(u_{2,2} + u_{4,4} + u_{2,4} + u_{4,2})$$
$$u_{1,1} = \frac{1}{4}(u_{0,0} + u_{2,2} + u_{0,2} + u_{2,0})$$
$$u_{3,1} = \frac{1}{4}(u_{2,0} + u_{4,2} + u_{2,2} + u_{4,0}).$$

In the second step, the remaining values, viz., $u_{2,3}, u_{1,2}, u_{3,2}$ and $u_{2,1}$ are evaluated using standard five-point (blue meshes in the figure). Thus,

$$u_{2,3} = \frac{1}{4}(u_{1,3} + u_{3,3} + u_{2,2} + u_{2,4})$$
$$u_{1,2} = \frac{1}{4}(u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3})$$
$$u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,1} + u_{3,3})$$
$$u_{2,1} = \frac{1}{4}(u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2}).$$

Note that these are the first approximate values of u at different meshes. These values can be updated by using any iterative methods mentioned earlier.

Example 3.1 *Let us consider the following Dirichlet's problem*

$$u_{xx} + u_{yy} = 0,$$
$$u(x,0) = 0, \quad u(0,y) = 0,$$
$$u(x,1) = 5x, \quad u(1,y) = 5y.$$

Find the first approximate values at the interior meshes by dividing the square region into 4×4 squares.

Solution. For this problem, the region R is $0 \le x, y \le 1$. Let $h = k = 0.25$ and $x_i = ih, y_j = jk$, $i, j = 0, 1, 2, 3, 4$. The meshes are shown in Figure 3.5.

The values of u are calculated in two steps. In first step, the diagonal five-point formula is used to find the values of $u_{2,2}, u_{1,3}, u_{3,3}, u_{1,1}$, $u_{3,1}$ and in second step the standard five-point formula is used to find the values of $u_{2,3}, u_{1,2}, u_{3,2}$, $u_{2,1}$.

The diagonal five-point formula is

$$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}]$$

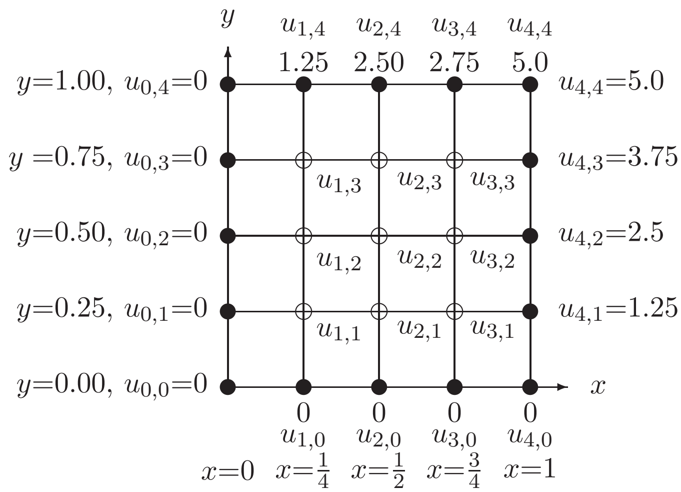


Figure 3.5: Meshes for Dirichlet's problem

and standard five-point formula is

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}].$$

Therefore,

$$\begin{aligned}
u_{2,2} &= \frac{1}{4}(u_{0,0} + u_{4,4} + u_{0,4} + u_{4,0}) = \frac{1}{4}(0 + 5.0 + 0 + 0) = 1.25 \\
u_{1,3} &= \frac{1}{4}(u_{0,2} + u_{2,4} + u_{0,4} + u_{2,2}) = \frac{1}{4}(0 + 2.5 + 0 + 1.25) = 0.9375 \\
u_{3,3} &= \frac{1}{4}(u_{2,2} + u_{4,4} + u_{2,4} + u_{4,2}) = \frac{1}{4}(1.25 + 5 + 2.5 + 2.5) = 2.8125 \\
u_{1,1} &= \frac{1}{4}(u_{0,0} + u_{2,2} + u_{0,2} + u_{2,0}) = \frac{1}{4}(0 + 1.25 + 0 + 0) = 0.3125 \\
u_{3,1} &= \frac{1}{4}(u_{2,0} + u_{4,2} + u_{2,2} + u_{4,0}) = \frac{1}{4}(0 + 2.5 + 1.25 + 0) = 0.9375.
\end{aligned}$$

The values of $u_{2,3}, u_{1,2}, u_{3,2}$ and $u_{2,1}$ are obtained by using standard five-point formula.

$$\begin{aligned}
u_{2,3} &= \frac{1}{4}(u_{1,3} + u_{3,3} + u_{2,2} + u_{2,4}) = \frac{1}{4}(0.9375 + 2.8125 + 1.25 + 2.5) = 1.875 \\
u_{1,2} &= \frac{1}{4}(u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3}) = \frac{1}{4}(0 + 1.25 + 0.3125 + 0.9375) = 0.625 \\
u_{3,2} &= \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,1} + u_{3,3}) = \frac{1}{4}(1.25 + 2.5 + 0.9375 + 2.8125) = 1.875 \\
u_{2,1} &= \frac{1}{4}(u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2}) = \frac{1}{4}(0.3125 + 0.9375 + 0 + 1.25) = 0.625.
\end{aligned}$$

These are the first approximate values of u at the interior meshes.

3.2 Iterative methods

If the first approximate values of u are known, then these values can be updated by applying any well known iterative method. Several iterative methods are available with different rates of convergence, some of them are discussed below.

The standard five-point finite-difference formula for the Poisson's equation (3.5) is

$$u_{i,j} = \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 g_{i,j}). \tag{3.8}$$

Let $u_{i,j}^{(r)}$ be the rth iterative value of $u_{i,j}$, $r = 1, 2, \ldots$.

Jacobi's method

The Jacobi's iterative scheme to solve the system of equations (3.8) for the interior meshes is

$$u_{i,j}^{(r+1)} = \frac{1}{4}[u_{i-1,j}^{(r)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r)} + u_{i,j+1}^{(r)} - h^2 g_{i,j}]. \tag{3.9}$$

This formula evaluates the $(r+1)$th iterated value of $u_{i,j}$, when the rth iterated values of u are known at the meshes $(i-1,j), (i+1,j), (i,j-1), (i,j+1)$.

Gauss-Seidel's method

In it well known (discussed in Chapter 5) that the latest updated values are used in Gauss-Seidel's method. The values of u along each row are computed systematically from left to right. The iterative formula is

$$u_{i,j}^{(r+1)} = \frac{1}{4}[u_{i-1,j}^{(r+1)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r+1)} + u_{i,j+1}^{(r)} - h^2 g_{i,j}]. \tag{3.10}$$

The rate of convergence of this method is twice as fast as the Jacobi's method.

Successive Over-Relaxation (SOR) method

In this method, the iteration scheme is accelerated by introducing a scalar, called relaxation factor. This acceleration is made by making corrections on $[u_{i,j}^{(r+1)} - u_{i,j}^{(r)}]$. Suppose $\overline{u_{i,j}^{(r+1)}}$ is the value obtained from any iteration method, such as Jacobi's or

Gauss-Seidel's method. Then the updated value of $u_{i,j}$ at the $(r+1)$th iteration is given by

$$u_{i,j}^{(r+1)} = w\overline{u_{i,j}^{(r+1)}} + (1-w)u_{i,j}^{(r)}, \tag{3.11}$$

where w is called relaxation factor.

If $w > 1$ then the method is called over-relaxation method. If $w = 1$ then the method is nothing but the Gauss-Seidal iteration method.

Thus, for the Poisson's equation, the Jacobi's over-relaxation scheme is

$$u_{i,j}^{(r+1)} = \frac{1}{4}w\Big[u_{i-1,j}^{(r)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r)} + u_{i,j+1}^{(r)} - h^2 g_{i,j}\Big] + (1-w)u_{i,j}^{(r)} \tag{3.12}$$

and the Gauss-Seidel's over-relaxation scheme is

$$u_{i,j}^{(r+1)} = \frac{1}{4}w\Big[u_{i-1,j}^{(r+1)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r+1)} + u_{i,j+1}^{(r)} - h^2 g_{i,j}\Big] + (1-w)u_{i,j}^{(r)}. \tag{3.13}$$

The rate of convergence of the above schema depends on the value of w and its value lies between 1 and 2. But, the choice of suitable w is a difficult task.

Example 3.2 *Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ defined within the square region $0 \le x \le 1$ and $0 \le y \le 1$ shown in Figure 3.6, by (a) Jacobi's method, (b) Gauss-Seidel's method, and (c) Gauss-Seidel's successive over-relaxation method.*

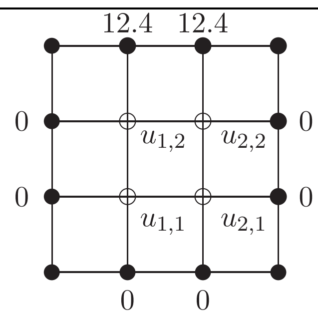


Figure 3.6: Boundary conditions of Laplace equation

Solution.
(a) ***Jacobi's method***

Let the initial values be $u_{2,1} = u_{1,2} = u_{2,2} = u_{1,1} = 0$ and $h = k = 1/3$.

The Jacobi's iteration scheme is

$$\begin{aligned}
u_{1,1}^{(r+1)} &= \frac{1}{4}\big[u_{2,1}^{(r)} + u_{1,2}^{(r)} + 0 + 0\big] = \frac{1}{4}\big[u_{2,1}^{(r)} + u_{1,2}^{(r)}\big] \\
u_{2,1}^{(r+1)} &= \frac{1}{4}\big[u_{1,1}^{(r)} + u_{2,2}^{(r)} + 0 + 0\big] = \frac{1}{4}\big[u_{1,1}^{(r)} + u_{2,2}^{(r)}\big] \\
u_{1,2}^{(r+1)} &= \frac{1}{4}\big[u_{1,1}^{(r)} + u_{2,2}^{(r)} + 0 + 12.4\big] = \frac{1}{4}\big[u_{1,1}^{(r)} + u_{2,2}^{(r)} + 12.4\big] \\
u_{2,2}^{(r+1)} &= \frac{1}{4}\big[u_{1,2}^{(r)} + u_{2,1}^{(r)} + 12.4 + 0\big] = \frac{1}{4}\big[u_{1,2}^{(r)} + u_{2,1}^{(r)} + 12.4\big].
\end{aligned}$$

The first iterated values are, $u_{1,1}^{(1)} = 0, u_{2,1}^{(1)} = 0, u_{1,2}^{(1)} = 3.1, u_{2,2}^{(1)} = 3.1$.

The all other iterated values are given below.

r	$u_{1,1}$	$u_{2,1}$	$u_{1,2}$	$u_{2,2}$
0	0.00000	0.00000	0.00000	0.00000
1	0.00000	0.00000	3.10000	3.10000
2	0.77500	0.77500	3.87500	3.87500
3	1.16250	1.16250	4.26250	4.26250
4	1.35625	1.35625	4.45625	4.45625
5	1.45312	1.45312	4.55312	4.55312
6	1.50156	1.50156	4.60156	4.60156
7	1.52578	1.52578	4.62578	4.62578
8	1.53789	1.53789	4.63789	4.63789
9	1.54395	1.54395	4.64395	4.64395
10	1.54697	1.54697	4.64697	4.64697
11	1.54849	1.54849	4.64849	4.64849
12	1.54924	1.54924	4.64924	4.64924
13	1.54962	1.54962	4.64962	4.64962
14	1.54981	1.54981	4.64981	4.64981
15	1.54991	1.54991	4.64991	4.64991
16	1.54995	1.54995	4.64995	4.64995

Therefore, $u_{1,1} = 1.5500$, $u_{2,1} = 1.5500$, $u_{1,2} = 4.6500$, $u_{2,2} = 4.6500$, correct up to four decimal places.

(b) ***Gauss-Seidel's method***

Let $u_{2,1} = u_{1,2} = u_{2,2} = u_{1,1} = 0$ be the initial values. Also, $h = k = 1/3$.

The Gauss-Seidel's iteration scheme is

$$\begin{aligned}
u_{1,1}^{(r+1)} &= \frac{1}{4}\left[u_{2,1}^{(r)} + u_{1,2}^{(r)}\right] \\
u_{2,1}^{(r+1)} &= \frac{1}{4}\left[u_{1,1}^{(r+1)} + u_{2,2}^{(r)}\right] \\
u_{1,2}^{(r+1)} &= \frac{1}{4}\left[u_{1,1}^{(r+1)} + u_{2,2}^{(r)} + 12.4\right] \\
u_{2,2}^{(r+1)} &= \frac{1}{4}\left[u_{1,2}^{(r+1)} + u_{2,1}^{(r+1)} + 12.4\right].
\end{aligned}$$

When $r = 0$ then

$$\begin{aligned}
u_{1,1}^{(1)} &= \frac{1}{4}\left[0+0\right] = 0 \\
u_{2,1}^{(1)} &= \frac{1}{4}\left[0+0\right] = 0 \\
u_{1,2}^{(1)} &= \frac{1}{4}\left[0+0+12.4\right] = 3.1 \\
u_{2,2}^{(1)} &= \frac{1}{4}\left[3.1+0+12.4\right] = 3.857.
\end{aligned}$$

These are the first iterated values. The results in all other iterations are shown below.

r	$u_{1,1}$	$u_{2,1}$	$u_{1,2}$	$u_{2,2}$
0	0.00000	0.00000	0.00000	0.00000
1	0.00000	0.00000	3.10000	3.87500
2	0.77500	1.16250	4.26250	4.45625
3	1.35625	1.45312	4.55312	4.60156
4	1.50156	1.52578	4.62578	4.63789
5	1.53789	1.54395	4.64395	4.64697
6	1.54697	1.54849	4.64849	4.64924
7	1.54924	1.54962	4.64962	4.64981
8	1.54981	1.54991	4.64991	4.64995
9	1.54995	1.54998	4.64998	4.64999
10	1.54999	1.54999	4.64999	4.65000
11	1.55000	1.55000	4.65000	4.65000

Hence, $u_{1,1} = 1.55000$, $u_{2,1} = 1.55000$, $u_{1,2} = 4.65000$, $u_{2,2} = 4.65000$, correct up to five decimal places.

(c) ***Gauss-Seidel's successive over-relaxation method***

Let the intial value be $u_{2,1} = u_{1,2} = u_{2,2} = u_{1,1} = 0$.

The SOR scheme for interior meshes are

$$u_{i,j}^{(r+1)} = \frac{w}{4}[u_{i-1,j}^{(r+1)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r+1)} + u_{i,j+1}^{(r)}] + (1-w)u_{i,j}^{(r)}.$$

For $j = 1, 2$, $i = 1, 2$, the formulae are

$$u_{1,1}^{(r+1)} = \frac{w}{4}[u_{2,1}^{(r)} + u_{1,2}^{(r)}] + (1-w)u_{1,1}^{(r)}$$
$$u_{2,1}^{(r+1)} = \frac{w}{4}[u_{1,1}^{(r+1)} + u_{2,2}^{(r)}] + (1-w)u_{2,1}^{(r)}$$
$$u_{1,2}^{(r+1)} = \frac{w}{4}[u_{2,2}^{(r)} + u_{1,1}^{(r+1)} + 12.4] + (1-w)u_{1,2}^{(r)}$$
$$u_{2,2}^{(r+1)} = \frac{w}{4}[u_{1,2}^{(r+1)} + u_{3,2}^{(r)} + u_{2,1}^{(r+1)} + 12.4] + (1-w)u_{2,2}^{(r)}.$$

Let $w = 1.1$. Then the values of u's are listed below.

r	$u_{1,1}$	$u_{2,1}$	$u_{1,2}$	$u_{2,2}$
1	0.00000	0.00000	3.41000	4.34775
2	0.93775	1.45351	4.52251	4.61863
3	1.54963	1.55092	4.65402	4.65450
4	1.55140	1.55153	4.65122	4.65031
5	1.55062	1.55010	4.65013	4.65003
6	1.55000	1.55000	4.65000	4.65000
7	1.55000	1.55000	4.65000	4.65000

Hence, solution is $u_{1,1} = 1.55000$, $u_{2,1} = 1.55000$, $u_{1,2} = 4.65000$, $u_{2,2} = 4.65000$, correct up to five decimal places.

The SOR iteration scheme gives the result in 6th iterations for $w = 1.1$. While Gauss-Seidal and Jacob's iteration schema take 11 and 16 iterations respectively.

For SOR method the number of iterations depends on the value of w.

Example 3.3 *Solve the Poisson's equation $u_{xx} + u_{yy} = 5x^2y$ for the square region $0 \le x \le 1, 0 \le y \le 1$ with $h = 1/3$ and the values of u on the boundary are every where zero. Use (a) Gauss-Seidel's method, and (b) Gauss-Seidel's SOR method.*

Solution. In this problem, $g(x, y) = 5x^2y$, $h = k = 1/3$ and the boundary conditions are $u_{0,0} = u_{1,0} = u_{2,0} = u_{3,0} = 0$, $u_{0,1} = u_{0,2} = u_{0,3} = 0$,
$u_{1,3} = u_{2,3} = u_{3,3} = 0, u_{3,1} = u_{3,2} = 0$.

(a) The Gauss-Seidel's iteration scheme is

$$u_{i,j}^{(r+1)} = \frac{1}{4}\big[u_{i-1,j}^{(r+1)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r+1)} + u_{i,j+1}^{(r)} - h^2 g(ih, jk)\big].$$

Now, $g(ih, jk) = 5h^3 i^2 j = \frac{5}{27} i^2 j$. Thus

$$\begin{aligned}
u_{1,1}^{(r+1)} &= \frac{1}{4}\left[u_{0,1}^{(r+1)} + u_{2,1}^{(r)} + u_{1,0}^{(r+1)} + u_{1,2}^{(r)} - \frac{1}{9}.\frac{5}{27}.1^2.1\right] \\
&= \frac{1}{4}\left[0 + u_{2,1}^{(r)} + 0 + u_{1,2}^{(r)} - \frac{5}{243}\right] = \frac{1}{4}\left[u_{2,1}^{(r)} + u_{1,2}^{(r)} - \frac{5}{243}\right] \\
u_{2,1}^{(r+1)} &= \frac{1}{4}\left[u_{1,1}^{(r+1)} + u_{3,1}^{(r)} + u_{2,0}^{(r+1)} + u_{2,2}^{(r)} - \frac{1}{9}.\frac{5}{27}.2^2.1\right] \\
&= \frac{1}{4}\left[u_{1,1}^{(r+1)} + u_{2,2}^{(r)} - \frac{20}{243}\right] \\
u_{1,2}^{(r+1)} &= \frac{1}{4}\left[u_{0,2}^{(r+1)} + u_{2,2}^{(r)} + u_{1,1}^{(r+1)} + u_{1,3}^{(r)} - \frac{1}{9}.\frac{5}{27}.1^2.2\right] \\
&= \frac{1}{4}\left[u_{2,2}^{(r)} + u_{1,1}^{(r+1)} - \frac{10}{243}\right] \\
u_{2,2}^{(r+1)} &= \frac{1}{4}\left[u_{1,2}^{(r+1)} + u_{3,2}^{(r)} + u_{2,1}^{(r+1)} + u_{2,3}^{(r)} - \frac{1}{9}.\frac{5}{27}.2^2.2\right] \\
&= \frac{1}{4}\left[u_{1,2}^{(r+1)} + u_{2,1}^{(r+1)} - \frac{40}{243}\right].
\end{aligned}$$

Let $u_{2,1}^{(0)} = u_{2,2}^{(0)} = u_{1,2}^{(0)} = 0$.

All the values are shown in the following table.

r	$u_{1,1}$	$u_{2,1}$	$u_{1,2}$	$u_{2,2}$
1	−0.00514	−0.02186	−0.01157	−0.04951
2	−0.01350	−0.03633	−0.02604	−0.05675
3	−0.02074	−0.03995	−0.02966	−0.05855
4	−0.02255	−0.04085	−0.03056	−0.05901
5	−0.02300	−0.04108	−0.03079	−0.05912
6	−0.02311	−0.04113	−0.03085	−0.05915
7	−0.02314	−0.04115	−0.03086	−0.05915

Hence, the solution correct up to five decimal places is

$u_{1,1} = -0.02314$, $u_{2,1} = -0.04115$, $u_{1,2} = -0.03086$, $u_{2,2} = -0.05915$.

. . . .

(b) The SOR scheme is

$$\begin{aligned} u_{i,j}^{(r+1)} &= \frac{w}{4}\left[u_{i-1,j}^{(r+1)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r+1)} + u_{i,j+1}^{(r)} - h^2 g(ih, jh)\right] + (1-w)u_{i,j}^{(r)} \\ &= \frac{w}{4}\left[u_{i-1,j}^{(r+1)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r+1)} + u_{i,j+1}^{(r)} - \frac{5}{243}i^2 j\right] + (1-w)u_{i,j}^{(r)}. \end{aligned}$$

Let the initial values be $u_{2,1}^{(0)} = u_{1,2}^{(0)} = u_{2,2}^{(0)} = 0$.

Let the relaxation factor be $w = 1.1$. Then, the values of $u_{1,1}, u_{2,1}, u_{1,2}$ and $u_{2,2}$ are computed below.

r	$u_{1,1}$	$u_{2,1}$	$u_{1,2}$	$u_{2,2}$
1	−0.00566	−0.02419	−0.01287	−0.05546
2	−0.01528	−0.03967	−0.02948	−0.05874
3	−0.02315	−0.04119	−0.03089	−0.05921
4	−0.02316	−0.04117	−0.03088	−0.05916
5	−0.02316	−0.04115	−0.03087	−0.05916
6	−0.02315	−0.04115	−0.03086	−0.05916

The solution obtained by SOR method is $u_{1,1} = -0.02315$, $u_{2,1} = -0.04115$, $u_{1,2} = -0.03086$, $u_{2,2} = -0.05916$, correct up to five decimal places.

Note that the Gauss-Seidel's iteration method needs 7 iterations whereas SOR method takes only 6 iteration for $w = 1.1$.

Self Assessment (MCQ)

1. The boundary conditions of the PDE $\nabla^2 u = 0$ are $u(x,0) = 0, u(0,y) = 0, u(x,1) = 2, u(1,y) = 2$, where $0 \leq x \leq 1$ and $0 \leq y \leq 1$, $u_{4,0} = 0, u_{0,4} = 0$. Let $h = k = 0.25$. Then the values of $u_{2,2}, u_{3,3}$ and $u_{1,1}$ are
   (a) $u_{2,2} = 0.50, u_{3,3} = 1.625, u_{1,1} = 0.125$
   (b) $u_{2,2} = 0.25, u_{3,3} = 0.625, u_{1,1} = 0.125$
   (c) $u_{2,2} = 0.75, u_{3,3} = 1.625, u_{1,1} = 1.125$
   (d) $u_{2,2} = 0.50, u_{3,3} = 1.505, u_{1,1} = 1.25$

2. For the elliptic PDE $u_{xx} + u_{yy} = 0$, the boundary conditions are $u_{i,0} = 60, i = 0,1,2,3$, $u_{0,1} = 40, u_{0,2} = 20, u_{0,3} = 0, u_{3,1} = 50, u_{3,2} = 40, u_{3,3} = 30, u_{1,3} = 10, u_{2,3} = 20$. Then the values of $u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}$ are
   (a) $u_{1,1} = 13.31, u_{1,2} = 23.60, u_{2,1} = 26.65, u_{2,2} = 33.32$
   (b) $u_{1,1} = 23.31, u_{1,2} = 22.60, u_{2,1} = 44.65, u_{2,2} = 21.02$
   (c) $u_{1,1} = 43.31, u_{1,2} = 26.65, u_{2,1} = 46.65, u_{2,2} = 33.32$
   (d) $u_{1,1} = 23.21, u_{1,2} = 22.35, u_{2,1} = 45.65, u_{2,2} = 32.22$

3. Successive over-relaxation method accelerates the convergence for all values of relaxation factor
   (a) true (b) false

4. Is diagonal five-point formula implicit?
   (a) yes (b) no

5. The iteration scheme $u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}]$ for $u_{xx} + u_{yy} = 0$ is known as standard five-point formula.
   (a) true (b) false

6. The iteration scheme $u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}]$ for $u_{xx} + u_{yy} = 0$ is known as diagonal five-point formula.
   (a) true (b) false

7. The finite difference iteration scheme $u_{i,j} = \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 g_{i,j})$

is nothing but a system of linear equations for different values of i and j. Then the Gauss-Seidal's iteration scheme is $\cdots\cdots$.

8. Finite-difference scheme for the Poisson's equation $u_{xx}+u_{yy} = g(x,y)$ is discretized as standard five-point formula is $\cdots\cdots$.

9. Let the Laplace equation be $u_{xx} + u_{yy} = 0$, defined by $0 \le x \le 1$ and $0 \le y \le 1$. Let $h = k = 1/3$. The boundary values are $u_{1,0} = u_{2,0} = u_{0,1} = u_{0,2} = 0$, $u_{1,3} = 5, u_{2,3} = 6$. The Jacobi's iteration scheme to find the values of $u_{1,1}, u_{2,1}, u_{1,2}$ and $u_{2,2}$ are $\cdots$.

10. For the above problem the Gauss-Seidal's iteration scheme is $\cdots$.

Answer to the questions

1. (a)
2. (c)
3. (b)
4. (b)
5. (d)
6. (a)
7. (a)
8. $u_{i,j}^{(r+1)} = \frac{1}{4}[u_{i-1,j}^{(r+1)} + u_{i+1,j}^{(r)} + u_{i,j-1}^{(r+1)} + u_{i,j+1}^{(r)} - h^2 g_{i,j}]$

5. $u_{i,j} = \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 g_{i,j})$.

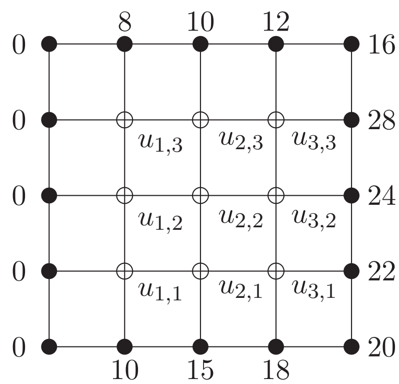
9. $u_{1,1}^{(r+1)} = \frac{1}{4}[u_{2,1}^{(r)} + u_{1,2}^{(r)}]$, $u_{2,1}^{(r+1)} = \frac{1}{4}[u_{1,1}^{(r)} + u_{2,2}^{(r)}]$,
   $u_{1,2}^{(r+1)} = \frac{1}{4}[u_{1,1}^{(r)} + u_{2,2}^{(r)} + 5]$, $u_{2,2}^{(r+1)} = \frac{1}{4}[u_{1,2}^{(r)} + u_{2,1}^{(r)} + 6]$.

10. $u_{1,1}^{(r+1)} = \frac{1}{4}[u_{2,1}^{(r)} + u_{1,2}^{(r)}]$, $u_{2,1}^{(r+1)} = \frac{1}{4}[u_{1,1}^{(r+1)} + u_{2,2}^{(r)}]$,
    $u_{1,2}^{(r+1)} = \frac{1}{4}[u_{1,1}^{(r+1)} + u_{2,2}^{(r)} + 5]$, $u_{2,2}^{(r+1)} = \frac{1}{4}[u_{1,2}^{(r+1)} + u_{2,1}^{(r+1)} + 6]$.

Self Assessment (Long Answer Questions)

1. Solve the Poisson's equation $u_{xx} + u_{yy} = -2x^2 + y^2$ over the region $0 \leq x \leq 2, 0 \leq y \leq 2$ taking the boundary condition $u = 0$ on all the boundary sides with $h = 0.5$. Use Gauss-Seidel's method to improve the solution.

2. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ taking $h = 1$, with boundary values as shown below.



3. Solve the elliptic differential equation $u_{xx} + u_{yy} = 0$ and for the region bounded by $0 \leq x \leq 5$, $0 \leq y \leq 5$, the boundary conditions being
   $u = 0$ at $x = 0$ and $u = 2 + y$ at $x = 5$,
   $u = x^2$ at $y = 0$ and $u = 2x$ at $y = 5$.

   Take $h = k = 1$. Use
   (a) Jacobi's method, (b) Gauss-Seidel's method, and (c) Gauss-Seidel's S.O.R. method.

Learn More

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