M.Sc. Course in
Applied Mathematics
with Oceanology and Computer Programming
Vidyasagar University

Semester-II

Paper-MTM 202 **Paper Name: Numerical Analysis**

Numerical Solution of Partial Differential Equations

Module No. 2

Partial Differential Equation: Hyperbolic

Objective

(a) Finite difference approximation of Hyperbolic PDEs

(b) Solution of wave equation

(c) Explicit method to solve Hyperbolic PDEs

(d) Implicit method to solve Hyperbolic PDEs

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In this module, two other types of partial differential equations are considered. These are hyperbolic and elliptic PDEs. Finite difference method is also used to solve these problems. First we consider hyperbolic equation.

2.1 Hyperbolic equations

The simplest problem of this class is one dimensional wave equation. This problem may occurs in many real life situations. For example, the transverse vibration of a stretched string, propagation of light and sound, propagation of water wave, etc. It arises in different fields such as acoustics, electromagnetics, fluid dynamics, etc.

The simplest form of the wave equation is given below:
Let $u = u(\mathbf{r}, t), \mathbf{r} \in R^n$ be a scalar function which satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{2.1}$$

where ∇^2 is the Laplacian in R^n and c is a constant speed of the wave propagation. This equation can also be written as

$$\Box^2 u = 0, \text{ where } \Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{2.2}$$

The operator $\Box^2$ is called d'Alembertian.

In case of one dimension, the above equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < 1. \tag{2.3}$$

The initial conditions are $u(x, 0) = f(x)$ and

$$\left(\frac{\partial u}{\partial t}\right)_{(x,0)} = g(x), 0 < x < 1 \tag{2.4}$$

and the boundary conditions are

$$u(0, t) = \phi(t) \text{ and } u(1, t) = \psi(t), t \geq 0. \tag{2.5}$$

In finite difference method, the partial derivatives u_{xx} and u_{tt} are approximated by the following central-difference schema at the mesh points $(x_i, t_j) = (ih, jk)$ are

$$u_{xx} = \frac{1}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + O(h^2)$$
$$\text{and } u_{tt} = \frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) + O(k^2),$$

where $i, j = 0, 1, 2, \ldots$.

Using this approximation, the equation (2.3) reduces to

$$\frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) = \frac{c^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}).$$

That is,

$$u_{i,j+1} = r^2 u_{i-1,j} + 2(1 - r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1}, \tag{2.6}$$

where $r = ck/h$.

Note that the value of $u_{i,j+1}$ depends on the values of u at two time-levels $(j-1)$, j and the value of $u_{i,j+1}$ can be determined if the four values $u_{i-1,j}, u_{i,j}, u_{i+1,j}, u_{i,j-1}$ are known.

The known (filled circle) and unknown (circle) values of u are shown in Figure 2.1.

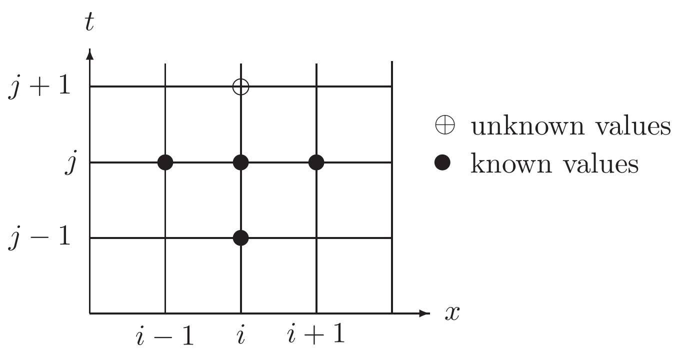

Figure 2.1: Known and unknown meshes for hyperbolic equations.

When $j = 0$, then from the equation (2.6) we get

$$u_{i,1} = r^2 u_{i-1,0} + 2(1 - r^2)u_{i,0} + r^2 u_{i+1,0} - u_{i,-1}.$$

Since, $u(x, 0) = f(x)$, $u_{i,0} = f(x_i) = f_i$. Using this notation, the above equation reduces to

$$u_{i,1} = r^2 f_{i-1} + 2(1 - r^2) f_i + r^2 f_{i+1} - u_{i,-1}. \tag{2.7}$$

Now, by central difference approximation, the initial condition (2.4), becomes

$$\frac{1}{2k}(u_{i,1} - u_{i,-1}) = g_i.$$

Substituting the value of $u_{i,-1}$ to the equation (2.7), we get

$$u_{i,1} = \frac{1}{2}\big[r^2 f_{i-1} + 2(1-r^2)f_i + r^2 f_{i+1} + 2kg_i\big]. \tag{2.8}$$

Thus, from equation (2.8) we obtain the values of $u_{i,1}$ for all values of i.

The truncation error of this method is $O(h^2+k^2)$ and the formula (2.6) is convergent for $0 < r \leq 1$.

Example 2.1 *Consider the following wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

The boundary conditions $u(0,t) = 0, u(1,t) = 0,\ t > 0$ *and the initial conditions* $u(x,0) = 4x^2, \left(\frac{\partial u}{\partial t}\right)_{(x,0)} = 0,\ 0 \leq x \leq 1$. *Find the value of* u *for* $x = 0, 0.2, 0.4, \ldots, 1.0$ *and* $t = 0, 0.1, 0.2, \ldots, 0.5,$ *when* $c = 1$.

Solution. Using central-difference approximation, the explicit formula for the given equation is

$$u_{i,j+1} = r^2 u_{i-1,j} + 2(1-r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1}. \tag{2.9}$$

Let $h = 0.2$ and $k = 0.1$, so $r = ck/h = 0.5 < 1$.

The boundary conditions transferred to $u_{0,j} = 0, u_{5,j} = 0$. The initial conditions reduce to $u_{i,0} = 4x_i^2, i = 1,2,3,4,5$ and $\dfrac{u_{i,1} - u_{i,-1}}{2k} = 0$, therefore $u_{i,-1} = u_{i,1}$.

Since $r = 0.5$, the difference equation (2.9) becomes

$$u_{i,j+1} = 0.25u_{i-1,j} + 1.5u_{i,j} + 0.25u_{i+1,j} - u_{i,j-1}. \tag{2.10}$$

When $j = 0$, then

$$\begin{aligned} u_{i,1} &= 0.25u_{i-1,0} + 1.5u_{i,0} + 0.25u_{i+1,0} - u_{i,-1} \\ \text{i.e. } u_{i,1} &= 0.125u_{i-1,0} + 0.75u_{i,0} + 0.125u_{i+1,0}, [\text{using } u_{i,-1} = u_{i,1}] \\ &= 0.125(u_{i-1,0} + u_{i+1,0}) + 0.75u_{i,0}. \end{aligned}$$

This formula gives the values of u for $j = 1$. For other values of j ($j = 2, 3, \ldots$) the values of u are calculated from the formula (2.10).

The initial and boundary values are shown in the following table.

$j=5, t=0.5$	0					0
$j=4, t=0.4$	0					0
$j=3, t=0.3$	0					0
$j=2, t=0.2$	0					0
$j=1, t=0.1$	0					0
$j=0, t=0.0$	0	0.16	0.64	1.44	2.56	0
	$x=0$	$x=0.2$	$x=0.4$	$x=0.6$	$x=0.8$	$x=1.0$
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$

The values of first row, i.e. $u_{i,1}$, $i=1,2,3,4$ are calculated as follows:

$$u_{1,1} = 0.125(u_{0,0}+u_{2,0}) + 0.75u_{1,0} = 0.20$$
$$u_{2,1} = 0.125(u_{1,0}+u_{3,0}) + 0.75u_{2,0} = 0.68$$
$$u_{3,1} = 0.125(u_{2,0}+u_{4,0}) + 0.75u_{3,0} = 1.48$$
$$u_{4,1} = 0.125(u_{3,0}+u_{5,0}) + 0.75u_{4,0} = 2.10.$$

Other values are written in the following table.

$j=5, t=0.5$	0	0.74328	0.89226	−0.50500	−1.25125	0
$j=4, t=0.4$	0	0.62906	1.05875	0.45250	−1.10375	0
$j=3, t=0.3$	0	0.46500	0.96625	1.17250	−0.29125	0
$j=2, t=0.2$	0	0.31000	0.80000	1.47500	0.96000	0
$j=1, t=0.1$	0	0.20000	0.68000	1.48000	2.10000	0
$j=0, t=0.0$	0	0.16000	0.64000	1.44000	2.56000	0
	$x=0$	$x=0.2$	$x=0.4$	$x=0.6$	$x=0.8$	$x=1.0$
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$

2.1.1 Implicit difference methods

Generally, implicit methods generate a tri-diagonal system of algebraic equations. Thus, it is suggested that the implicit methods should not be used without simplifying assumption to solve pure BVPs, because these methods generate large number of equations for small h and k. But, these methods may be used for initial-boundary value problems. Two such implicit methods are described below.

Implicit Method-I

The right hand side of the equation (2.3) is divided into two parts. Now, by central-difference approximation at the mesh point (ih, jk) the given equation reduces to

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{c^2}{2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j+1} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j-1}\right].$$

That is,

$$\begin{aligned}&\frac{1}{k^2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] \\ &= \frac{c^2}{2h^2}[(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}) + (u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1})].\end{aligned} \tag{2.11}$$

Implicit Method-II

Again, we divide the right hand side of the given equation into three parts as

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{c^2}{4}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j+1} + 2\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j-1}\right].$$

By central-difference approximation the given equation reduces to

$$\begin{aligned}&\frac{1}{k^2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] \\ &= \frac{c^2}{4h^2}[(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}) \\ &\quad +2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + (u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1})].\end{aligned} \tag{2.12}$$

The above equation can be written as

$$\begin{aligned}&-u_{i-1,j+1} + 2\Big(1 + \frac{2}{r^2}\Big)u_{i,j+1} - u_{i+1,j+1} \\ &\qquad = 2\Big[u_{i-1,j} - 2\Big(1 - \frac{2}{r^2}\Big)u_{i,j} + u_{i+1,j}\Big] \\ &\qquad\quad + \Big[u_{i-1,j-1} - 2\Big(1 + \frac{2}{r^2}\Big)u_{i,j-1} + u_{i+1,j-1}\Big]\end{aligned} \tag{2.13}$$

where $r = ck/h$.

This is a system of linear tri-diagonal equations and it can be solved by any method. The above system of equations can also be written as

$$\begin{bmatrix} 2s & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2s & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2s & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2s & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2s \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \\ \cdots \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} d_{1,j+1} \\ d_{2,j+1} \\ d_{3,j+1} \\ d_{4,j+1} \\ \cdots \\ d_{N-1,j+1} \end{bmatrix}$$

where $s = 1 + 2/r^2$ and

$$\begin{aligned} d_{1,j} &= u_{0,j+1} + 2[u_{0,j} - 2(1-2/r^2)u_{1,j} + u_{2,j}] + [u_{0,j-1} - 2(1+2/r^2)u_{1,j-1} + u_{2,j-1}] \\ d_{i,j} &= 2[u_{i-1,j} - 2(1-2/r^2)u_{i,j} + u_{i+1,j}] \\ &\quad + [u_{i-1,j-1} - 2(1+2/r^2)u_{i,j-1} + u_{i+1,j-1}] \\ &\quad i = 2, 3, \ldots, N-2 \\ d_{N-1,j} &= u_{N,j+1} + 2[u_{N-2,j} - 2(1-2/r^2)u_{N-1,j} + u_{N,j}] \\ &\quad + [u_{N-2,j-1} - 2(1+2/r^2)u_{N-1,j-1} + u_{N,j-1}] \end{aligned}$$

For a particular value of $j = k$, $k = 1, 2, \ldots$, one can find all values of $u_{i,k}$, for $i = 1, 2, \ldots, N-1$.

Both the formulae are valid for all values of $r = ck/h > 0$.

Self Assessment (MCQ)

1. Suppose $\frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$, $x_i = ih, y_j = jk$ be the iteration scheme to solve the PDE $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$. This iteration scheme will be stable, if

   (a) $1 < k/h \leq 2$

   (b) $0 < k/h \leq 0.5$

   (c) $0 < k/h \leq 1.5$

   (d) $0 < k/h \leq 1$

2. The truncation error of finite difference scheme used to solve hyperbolic equation is

   (a) $(h + k)$ (b) $(h^2 + k^2)$

   (a) $(h^3 + k^3)$ (d) $(h + k^2)$

3. The function $u(x, 0) = e^{\pi x}$ is discretized as

   (a) $u_{i,j} = e^{\pi ih}$ (b) $u_{i,j} = e^{\pi ijh}$

   (c) $u_i = e^{\pi ih}$ (d) $u_{i,0} = e^{\pi ih}$

4. The central difference approximation of $u_t(x, 0) = 0$ is

   (a) $\frac{u_{i,1}-u_{i,-1}}{2k} = 0$ (b) $\frac{u_{i,j}-u_{i,j-1}}{2k} = 0$

   (c) $\frac{u_{1,i}-u_{-1,i}}{2k} = 0$ (d) $\frac{u_{i,1}-u_{i,-1}}{k} = 0$

5. In explicit method to solve the wave equation $u_{tt} = c^2 u_{xx}$, the number of values of u required to find the value of another u is

   (a) 3 (b) 2 (c) 4 (d) 5

6. The finite difference scheme to solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < 1$ with the initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t} = g(x), t = 0, 0 < x < 1$ and boundary conditions $u(0, t) = \phi(t)$ and $u(1, t) = \psi(t), t \geq 0$ are $\cdots\cdots$.

7. Let $u_{tt} = c^2 u_{xx}$ be the wave equation with boundary conditions $u(0, t) = 0, u(1, t) = 0, t > 0$ and initial conditions $u(x, 0) = 4x^2$, $u_t(x, 0) = 0, 0 \leq x \leq 1$. Then the value of $u_{i,1}$ for $i = 1, 2, 3, 4$ when $h = 0.2$ are $\cdots$.

8. Let the PDE be $u_{tt} = c^2 u_{xx}$ and $h = k = 0.2$. If $u_{2,3} = 0.20, u_{1,4} = 0.25, u_{2,4} = 0.80, u_{3,4} = 0.60$ then the value of $u_{2,5}$ is $\cdots$.

9. The range of ck/h for the stable solutions of the wave equation $u_{tt} = c^2 u_{xx}$ by implicit method is $\cdots$.

Answer to the questions

1. (d)
2. (b)
3. (d)
4. (a)
5. (c)
6. $u_{i,j+1} = r^2 u_{i-1,j} + 2(1 - r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1}$, and
$u_{i,1} = r^2 u_{i-1,0} + 2(1 - r^2)u_{i,0} + r^2 u_{i+1,0} - u_{i,-1}$
$= r^2 f_{i-1} + 2(1 - r^2)f_i + r^2 f_{i+1} - u_{i,-1}$, where $r = ck/h$, $f_i = f(x_i)$
7. $u_{1,1} = 0.20, u_{2,1} = 0.68, u_{3,1} = 1.48, u_{4,1} = 2.10$.
8. $u_{2,5} = 0.65$.
9. $ck/h > 0$.

Self Assessment (Long Answer Questions)

1. The differential equation $u_{tt} = u_{xx}, 0 \le x \le 1$ satisfies the boundary conditions $u = 0$ at $x = 0$ and $x = 1$ for $t > 0$, and the initial conditions $u(x, 0) = \sin \frac{\pi x}{4}$, $\left(\frac{\partial u}{\partial t}\right)_{(x,0)} = 0$. Compute the values of u for $x = 0, 0.1, 0.2, \ldots, 0.5$ and $t = 0, 0.1, 0.2, \ldots, 0.5$.

2. Solve the hyperbolic partial differential equation $u_{tt} = u_{xx}, 0 \le x \le 2, t \ge 0$, subject to the boundary conditions $u(0, t) = u(2, t) = 0, t \ge 0$ and the initial conditions $u(x, 0) = 5 \sin \frac{\pi x}{2}, 0 \le x \le 2, u_t(x, 0) = 0$, taking $h = 1/8$ and $k = 1/8$.

Learn More

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