M.Sc. Course in
Applied Mathematics
with Oceanology and Computer Programming
Vidyasagar University

Semester-II

Paper-MTM 202 **Paper Name: Numerical Analysis**

Numerical Solution of Partial Differential Equations

Module No. 1

Partial Differential Equation: Parabolic

Objective

(a) Partial differential equations

(b) Finite difference schemes for PDEs

(c) Parabolic PDEs

(d) Crank-Nicolson method to solve parabolic PDE

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The partial differential equation (PDE) is one of the most important and useful topics of mathematics, physics and different branches of engineering. But, finding of solution of PDEs is a very difficult task. Several analytical methods are available to solve PDEs, but, all these methods need in depth mathematical knowledge. On the other hand numerical methods are simple, but generate erroneous result. Most widely used numerical method is finite-difference method due to its simplicity. In this module, only finite-difference method is discussed to solve PDEs.

1.1 Classification of partial differential equations

The general second order PDE is of the following form:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G \tag{1.1}$$

$$i.e., \qquad Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G, \tag{1.2}$$

where A, B, C, D, E, F, G are functions of x and y.

The second order PDEs are of three types, viz. elliptic, hyperbolic and parabolic. The type of a PDE can be determined by finding the discriminant

$$\Delta = B^2 - 4AC.$$

The PDE of equation (1.2) is called elliptic, parabolic and hyperbolic according as the value of Δ at any point (x, y) is $< 0, = 0$ or > 0.

Elliptic equation

The most simple examples of this type of PDEs are Poisson's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=g(x,y) \tag{1.3}$$

and Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{or} \qquad \nabla^2 u=0. \tag{1.4}$$

The analytic solution of an elliptic equation is a function of x and y which satisfies the PDE at every point of the region S which is bounded by a plane closed curve C and satisfies some conditions at every point on C. The condition that the dependent variable satisfies along the boundary curve C is known as **boundary condition**.

Parabolic equation

The simplest example of parabolic equation is the heat conduction equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}. \tag{1.5}$$

In parabolic PDE, time t is involved as an independent variable.

The solution u represents the temperature at a distance x units from one end of a thermally insulated bar after t seconds of heat conduction. In this problem, the temperature at the ends of a bar are known for all time, i.e. the boundary conditions are known.

Hyperbolic equation

The third type PDE is hyperbolic. Also, in this equation time t is taken as an independent variable.

The simplest example of hyperbolic equation is the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{1.6}$$

Here, u is the transverse displacement of a vibrating string of length l at a distance x from one end after a time t.

Hyperbolic equations generally originate from vibration problems.

The values of u at the ends of the string are generally known for all time (i.e. boundary conditions are known) and the shape and velocity of the string are given at initial time (i.e. initial conditions are known).

1.2 Finite-difference approximation to partial derivatives

Let x and y be two independent variables and u be the dependent variable of the given PDE. Now, we divide the xy-plane into set of equal rectangles of lengths $\Delta x = h$ and $\Delta y = k$ by drawing the equally spaced grid lines parallel to the coordinates axes. That is,

$$\begin{aligned} x_i &= ih, \qquad i = 0, \pm 1, \pm 2, \ldots \\ y_j &= jk, \qquad j = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

The intersection of horizontal and vertical lines is called mesh point and the ijth mesh point is denoted by $P(x_i, y_j)$ or $P(ih, jk)$. The value of u at this mesh point is denoted by $u_{i,j}$, i.e. $u_{i,j} = u(x_i, y_j) = u(ih, jk)$.

The first order partial derivatives at this mesh point are approximated as follows:

$$
\begin{aligned}
u_x(ih, jk) &= \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) && (1.7)\\
&\quad \text{(forward difference approximation)}\\
&= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) && (1.8)\\
&\quad \text{(backward difference approximation)}\\
&= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) && (1.9)\\
&\quad \text{(central difference approximation)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
u_y(ih, jk) &= \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) && (1.10)\\
&\quad \text{(forward difference approximation)}\\
&= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) && (1.11)\\
&\quad \text{(backward difference approximation)}\\
&= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) && (1.12)\\
&\quad \text{(central difference approximation)}
\end{aligned}
$$

The second order partial derivatives are approximated as follows:

$$
\begin{aligned}
u_{xx}(ih, jk) &= \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2). && (1.13)\\
u_{yy}(ih, jk) &= \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2). && (1.14)
\end{aligned}
$$

The above equations are used to approximate a PDE to a system of difference equations.

1.3 Parabolic equations

Here, two methods are described to solve a parabolic PDE.

Let us consider the following parabolic equation known as heat conduction equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1.15}$$

with the initial condition $u(x,0) = f(x)$ and the boundary conditions $u(0,t) = \phi(t), u(1,t) = \psi(t)$.

1.3.1 An explicit method

We approximate the given PDE by using the finite-difference approximation for u_t and u_{xx} defined in (1.10) and (1.13). Then the equation (1.15) reduces to

$$\frac{u_{i,j+1} - u_{i,j}}{k} \simeq \alpha \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \tag{1.16}$$

where $x_i = ih$ and $t_j = jk$; $i, j = 0, 1, 2, \ldots$.

After simplification the above equation becomes

$$u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + ru_{i+1,j}, \tag{1.17}$$

where $r = \alpha k/h^2$.

From this formula one can compute the value of $u_{i,j+1}$ at the mesh point $(i, j+1)$ when the values of $u_{i-1,j}, u_{i,j}$ and $u_{i+1,j}$ are known. So this method is called the explicit method. By stability analysis, it can be shown that the formula is stable, if $0 < r \le 1/2$.

The grids and mesh points are shown in Figure 1.1. For this problem, the initial and boundary values are given. These values are shown in the figure by filled circles. That is, the values of u are known along x-axis and two vertical lines ($t = 0$ and $t = 1$). Also, it is mentioned in the figure that if the values at the meshes $(i-1, j), (i, j)$ and $(i+1, j)$ are known (shown by filled circles) then one can determine the value of u for the mesh $(i, j+1)$ (shown by circle).

Example 1.1 *Solve the following heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(0,t) = 0, u(1,t) = 3t$ *and initial condition* $u(x,0) = 1.5x$.

Solution. Let $h = 0.2$ and $k = 0.01$, so $r = k/h^2 = 0.25 < 1/2$. The initial and boundary values are shown in the following table.

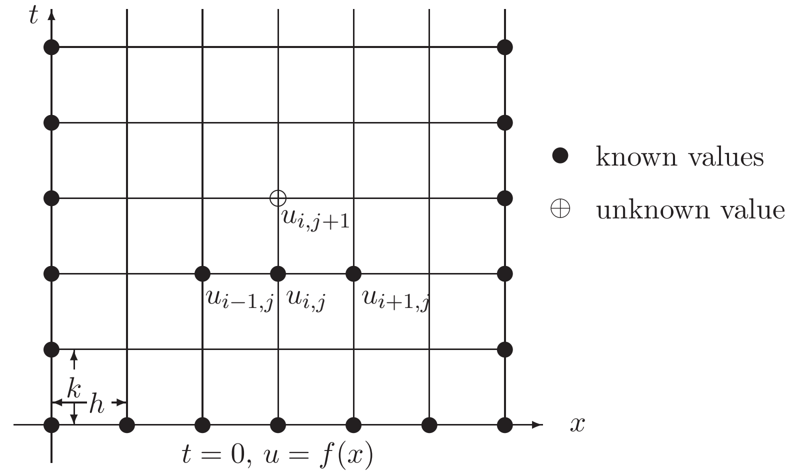

Figure 1.1: Known and unknown meshes of heat equation

$j = 5, t = 0.05$	0.0					0.15
$j = 4, t = 0.04$	0.0					0.12
$j = 3, t = 0.03$	0.0					0.09
$j = 2, t = 0.02$	0.0					0.06
$j = 1, t = 0.01$	0.0					0.03
$j = 0, t = 0.00$	0.0	0.3	0.6	0.9	1.2	0.00
	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$

Using the finite difference approximations $\dfrac{\partial u}{\partial t} = \dfrac{u_{i,j+1} - u_{i,j}}{k}$ and $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$, the given PDE reduces to

$$u_{i,j+1} = \frac{1}{4}(u_{i-1,j} + 2u_{i,j} + u_{i+1,j}).$$

From this equation, we get

$$u_{1,1} = \frac{1}{4}(u_{0,0} + 2u_{1,0} + u_{2,0}) = 0.3$$
$$u_{2,1} = \frac{1}{4}(u_{1,0} + 2u_{2,0} + u_{3,0}) = 0.6$$

and so on.

The values of u for all meshes are shown in the following table.

$j=5, t=0.05$	0.0	0.28104	0.50125	0.56968	0.42396	0.15
$j=4, t=0.04$	0.0	0.29264	0.53888	0.63460	0.47062	0.12
$j=3, t=0.03$	0.0	0.30000	0.57656	0.71438	0.53906	0.09
$j=2, t=0.02$	0.0	0.30000	0.60000	0.80625	0.64500	0.06
$j=1, t=0.01$	0.0	0.30000	0.60000	0.90000	0.82500	0.03
$j=0, t=0.00$	0.0	0.30000	0.60000	0.90000	1.20000	0.00
	$x=0$	$x=0.2$	$x=0.4$	$x=0.6$	$x=0.8$	$x=1.0$
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$

1.3.2 Crank-Nicolson implicit method

The above explicit method is very simple and it has limitation. This method is stable if $0 < r \le 1/2$, i.e. $0 < \alpha k/h^2 \le 1/2$, or $\alpha k \le h^2/2$. That is, the value of k must be chosen very small, and it takes time to get the result at a particular mesh. In 1947, Crank and Nicolson have developed an implicit method which reduces the total computation time. Also, the method is applicable for all finite values of r. In this method, the given PDE is approximated by replacing both space and time derivatives by their central difference approximations at the midpoint of the points (ih, jk) and $(ih, (j+1)k)$, i.e. at the point $(ih, (j+1/2)k)$. To use approximation at this point, we write the equation (1.15) in the following form

$$\left(\frac{\partial u}{\partial t}\right)_{i,j+1/2} = \alpha\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j+1/2} = \frac{\alpha}{2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j+1}\right]. \tag{1.18}$$

Then using central difference approximation the above equation reduces to

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{\alpha}{2}\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right].$$

After simplification, we get the following equation

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j}, \tag{1.19}$$

where $r = \alpha k/h^2$ and $j = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, (N-1)$, N is the number of subdivisions of x.

In general, the left hand side of the above equation contains three unknowns values, but the right hand side has three known values of u.

The known (circle) and unknown (filled circle) meshes are shown in Figure 1.2.

For $j = 0$ and $i = 1, 2, \ldots, N-1$, equation (1.19) generates N simultaneous equations for $N-1$ unknown $u_{1,1}, u_{2,1}, \ldots, u_{N-1,1}$ (of first row) in terms of known initial values $u_{1,0}, u_{2,0}, \ldots, u_{N-1,0}$ and boundary values $u_{0,0}$ and $u_{N,0}$. Thus, for this problem initial and boundary conditions are required.

Similarly, for $j = 1$ and $i = 1, 2, \ldots, N-1$ we obtain another system of equations containing the unknowns $u_{1,2}, u_{2,2}, \ldots, u_{N-1,2}$ in terms of known values obtained in previous step.

Thus for each value of j ($j = 2, 3, \ldots$), there is a system of equations.

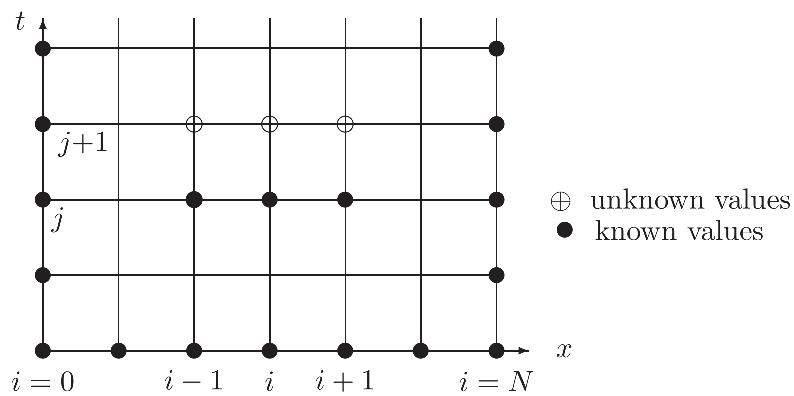

Figure 1.2: Meshes of Crank-Nicolson implicit method

In this method, the value of $u_{i,j+1}$ is not expressed directly in terms of known values of u's obtained in earlier step, it is written as unknown values, and hence the method is implicit.

The system of equations (1.19) for a fixed j, can be expressed as the following matrix notation.

$$\begin{bmatrix} 2{+}2r & -r & & & & & \\ & -r & 2{+}2r & -r & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & -r & 2{+}2r & -r \\ & & & & & -r & 2{+}2r \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} d_{1,j} \\ d_{2,j} \\ d_{3,j} \\ \vdots \\ d_{N-1,j} \end{bmatrix} \tag{1.20}$$

where

$$\begin{aligned} d_{1,j} &= ru_{0,j} + (2-2r)u_{1,j} + ru_{2,j} + ru_{0,j+1} \\ d_{i,j} &= ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j}; \qquad i = 2, 3, \ldots, N-2 \\ d_{N-1,j} &= ru_{N-2,j} + (2-2r)u_{N-1,j} + ru_{N,j} + ru_{N,j+1}. \end{aligned}$$

Note that the right hand side of equation (1.20) is known.

This is a tri-diagonal system of linear equations and it can be solved by any method discussed in Chapter 5 or the special method for tri-diagonal equations.

Example 1.2 *Solve the following problem by Crank-Nicolson method by taking $h = \frac{1}{4}$ and $k = \frac{1}{8}, \frac{1}{16}$*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, t > 0$$

where $u(0,t) = u(1,t) = 0, t > 0, u(x,0) = 3x, t = 0.$

Solution. ***Case I.***
Let $h = \frac{1}{4}$ and $k = \frac{1}{8}$. Therefore, $r = \frac{k}{h^2} = 2$.

The Crank-Nicolson scheme is

$$-u_{i-1,j+1} + 3u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}.$$

The initial and boundary conditions are shown in Figure 1.3.

The initial values are $u_{0,0} = 0, u_{1,0} = 0.75, u_{2,0} = 1.5, u_{3,0} = 2.25, u_{4,0} = 3$ and the boundary values are $u_{0,1} = 0, u_{4,1} = 0$.

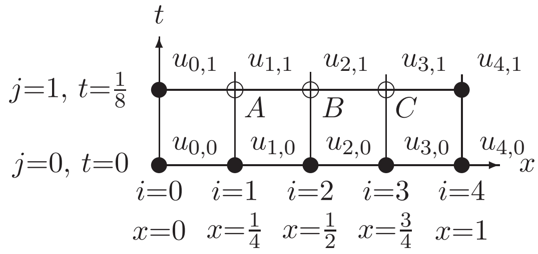

Figure 1.3: Boundary and initial values when $h = 1/4, k = 1/8$.

The unknown meshes are $A(i = 1, j = 1), B(i = 2, j = 1)$ and $C(i = 3, j = 1)$. Hence the system of equations is

$$-u_{0,1} + 3u_{1,1} - u_{2,1} = u_{0,0} - u_{1,0} + u_{2,0}$$
$$-u_{1,1} + 3u_{2,1} - u_{3,1} = u_{1,0} - u_{2,0} + u_{3,0}$$
$$-u_{2,1} + 3u_{3,1} - u_{4,1} = u_{2,0} - u_{3,0} + u_{4,0}.$$

Using initial and boundary values the above system of equations becomes

$$0 + 3u_{1,1} - u_{2,1} = 0 - 0.75 + 1.5 = 0.75$$
$$-u_{1,1} + 3u_{2,1} - u_{3,1} = 0.75 - 1.5 + 2.25 = 1.5$$
$$-u_{2,1} + 3u_{3,1} + 0 = 1.5 - 2.25 + 3.0 = 2.25.$$

This is a system of three linear equations with three unknowns. The solution of this system is
$u_{1,1} = u(0.25, 0.125) = 0.60714, \qquad u_{2,1} = u(0.50, 0.125) = 1.07143,$
$u_{3,1} = u(0.75, 0.125) = 1.10714.$

Case II.
Let $h = \dfrac{1}{4}, k = \dfrac{1}{16}$. Therefore, $r = 1$. To find the value of u at $t = \dfrac{1}{8}$, we have to solve two systems of tri-diagonal equations in two steps.

The Crank-Nicolson scheme is

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}.$$

For this case, the initial and boundary conditions are shown in Figure 1.4.

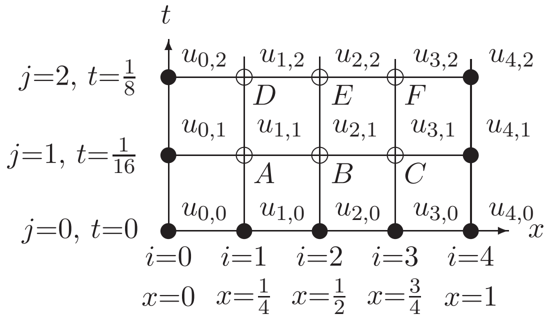

Figure 1.4: Boundary and initial values when $h = 1/4, k = 1/16$.

That is, the initial values are, $u_{0,0} = 0, u_{1,0} = 0.75, u_{2,0} = 1.5, u_{3,0} = 2.25, u_{4,0} = 3$ and the boundary values are $u_{0,1} = 0, u_{4,1} = 0; u_{0,2} = 0,\ u_{4,2} = 0$.

Here, $r = 1$, so the middle term of right hand side of Crank-Nicolson equation vanishes. Thus the Crank-Nicolson equations for first step, i.e. for the mesh points $A(i = 1, j = 1)$, $B(i = 2, j = 1)$ and $C(i = 3, j = 1)$ are respectively

$$-u_{0,1} + 4u_{1,1} - u_{2,1} = u_{0,0} + u_{2,0}$$
$$-u_{1,1} + 4u_{2,1} - u_{3,1} = u_{1,0} + u_{3,0}$$
$$-u_{2,1} + 4u_{3,1} - u_{4,1} = u_{2,0} + u_{4,0}.$$

That is,

$$4u_{1,1} - u_{2,1} = 0 + 1.5 = 1.5$$
$$-u_{1,1} + 4u_{2,1} - u_{3,1} = 0.75 + 2.25 = 3.0$$
$$-u_{2,1} + 4u_{3,1} = 1.5 + 3 = 4.5.$$

The solution of this system of equations is
$u_{1,1} = u(0.25, 0.0625) = 0.69643, u_{2,1} = u(0.50, 0.0625) = 1.28571,$
$u_{3,1} = u(0.75, 0.0625) = 1.44643.$

Again, the Crank-Nicolson equations for the mesh points
$D(i = 1, j = 2), E(i = 2, j = 2)$ and $F(i = 3, j = 2)$ are respectively,

$$-u_{0,2} + 4u_{1,2} - u_{2,2} = u_{0,1} + u_{2,1}$$
$$-u_{1,2} + 4u_{2,2} - u_{3,2} = u_{1,1} + u_{3,1}$$
$$-u_{2,2} + 4u_{3,2} - u_{4,2} = u_{2,1} + u_{4,1}.$$

. . . .

Using boundary conditions and values of right hand side obtained in first step, the above system becomes

$$\begin{aligned}
4u_{1,2} - u_{2,2} &= 0 + 1.28571 = 1.28571 \\
-u_{1,2} + 4u_{2,2} - u_{3,2} &= 0.69643 + 1.44643 = 2.14286 \\
-u_{2,2} + 4u_{3,2} \qquad &= 1.28571 + 0 = 1.28571.
\end{aligned}$$

The solution of this system is
$u_{1,2} = u(0.25, 0.125) = 0.52041, u_{2,2} = u(0.50, 0.125) = 0.79592,$
$u_{3,2} = u(0.75, 0.125) = 0.52041.$

Self Assessment (MCQ)

1. The PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y)$ is known as
   (a) Wave equation
   (b) Head equation
   (c) Poisson's equation
   (d) None of these

2. The wave equation is a
   (a) Parabolic PDE
   (b) Elliptic PDE
   (c) Hyperbolic PDE
   (d) None of these

3. The forward difference approximation of $u_y(x_i, y_j)$, where $x_i = x_0+ih, y_j = y_0+jk$ is
   (a) $\frac{u_{i,j}-u_{i,j-1}}{k} + O(k)$
   (b) $\frac{u_{i,j+1}-u_{i,j}}{k} + O(k)$
   (c) $\frac{u_{i,j+1}-u_{i,j-1}}{2k} + O(k^2)$
   (d) $\frac{u_{i,j}-u_{i-1,j}}{h} + O(h)$

4. The central difference approximation of $u_x(x_i, y_j)$, where $x_i = x_0+ih, y_j = y_0+jk$ is
   (a) $\frac{u_{i,j}-u_{i,j-1}}{k} + O(k)$
   (b) $\frac{u_{i,j+1}-u_{i,j-1}}{2k} + O(k^2)$
   (c) $\frac{u_{i+i,j}-u_{i-1,j}}{2h} + O(h^2)$
   (d) $\frac{u_{i,j}-u_{i-1,j}}{h} + O(h)$

5. To solve the heat equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ with the boundary condition $u(0, t) = f_1(t), u(1, t) = f_2(t)$ and initial condition $u(x, 0) = g(x)$ by explicit method
   (a) two values of u are required to get the next value
   (b) three values of u are required to get the next value
   (c) four values of u are required to get the next value
   (d) none of these

6. The iteration scheme $u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}$, where $r = \alpha k/h^2$ to solve the $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ is stable, if
   (a) $0.5 < r \leq 1.0$
   (b) $0 < r \leq 0.5$
   (c) $1 < r \leq 1.5$
   (d) for all values of r

7. Crank-Nicolson method converts a parabolic PDE into
   (a) a system of ordinary differential equation
   (b) a system of tri-diagonal linear equations
   (c) two first order PDEs
   (d) an explicit algebraic equation

8. Crank-Nicolson method, to solve a parabolic PDE, is an
   (a) explicit method (b) implicit method

9. The explicit iteration scheme $u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}$, where $r = \alpha k/h^2$ to solve the heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ gives more better result than Crank-Nicolson method.
   (a) true (b) false

10. The PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as wave equation.
   (a) true (b) false

11. Is $\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2)$, where $x_i = x_0 + ih, y_j = y_0 + jk$ the finite difference approximation of $u_{xx}(x_i, y_j)$?
   (a) yes (b) no

12. The forward difference approximation of the heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ is $\cdots\cdots$.

Answer to the questions

1. (c)
2. (c)
3. (b)
4. (c)

5. (b)
6. (b)
7. (b)
8. (b)
9. (b)
10. (b)
11. (a)
12. $u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}$, where $r = \alpha k/h^2$

Self Assessment (Long Answer Questions)

1. Solve the heat equation
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$
subject to the conditions $u(x,0) = 0, u(0,t) = 0$ and $u(1,t) = 2t$, taking $h = 1/2, k = 1/16$.

2. Given the differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ and the boundary condition $u(0,t) = u(5,t) = 0$ and $u(x,0) = x^2(30 - x^2)$. Use the explicit method to obtain the solution for $x_i = ih, y_j = jk$; $i = 0, 1, \ldots, 5$ and $j = 0, 1, 2, \ldots, 6$.

3. Solve the differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1/2$, given that $u = 0$ when $t = 0, 0 \le x \le 1/2$ and with boundary conditions $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $\frac{\partial u}{\partial x} = 1$ at $x = 1/2$ for $t > 0$, taking $h = 0.1, k = 0.001$.

4. Solve the following initial value problem $f_t = f_{xx}$, $0 \le x \le 1$ subject to the initial condition $f(x,0) = \cos\frac{\pi x}{2}$ and the boundary conditions $f(0,t) = 1, f(1,t) = 0$ for $t > 0$, taking $h = 1/3, k = 1/3$.

5. Solve the parabolic differential equation $u_t = u_{xx}, 0 \le x \le 1$ subject to the boundary conditions $u = 0$ at $x = 0$ and $x = 1$ for $t > 0$ and the initial conditions
$$u(x,0) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2(1-x) & \text{for } 1/2 \le x \le 1, \end{cases}$$
using explicit method.

Learn More

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