# M.Sc. Course
# in
# Applied Mathematics with Oceanology
# and
# Computer Programming

## Vidyasagar University

## Semestar-IV

Paper-MTM 403/II **Paper Name: Probability and Statistics**

Unit 2 Unit Name: **Markov Process**

***by***

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**Unit Structure:**



## 2.1 Introduction

In this unit, Markov process with discrete and continuous state space are introduced. Poisson process has many applications in real life situations where uncertainty occurs. Many real life problems can be dealt with birth and death process. Wiener process and branching process has also many applications.

Objectives:

Gone through this unit the readers will learn the following:
- Poisson process
- Pure birth process and its p.g.f
- Birth and death process, p.g.f, mean probability size, extinction probability
- Wiener process
- Branching process.

## 2.2 Discrete State Space: Poisson Process

Poisson process is a stochastic process in continuous time and discrete state space. Let $N(t)$ be the number of occurrences of the event $E$ in an interval $(0, t)$. Let $P_n(t)$ be the probability that the random variable $N(t)$ assumes the value $n$, i.e.,

$$p_n(t) = P(N(t) = n) \tag{2.1}$$

This probability is a function of time $t$. Since the only possible values of $n$ are $n = 0, 1, 2, \ldots$

$$\sum_{n=0}^{\infty} p_n(t) = 1 \tag{2.2}$$

Thus $\{p_n(t)\}$ represents the probability distribution of the random variable $N(t)$ for every value of $t$. The family of random variables $\{N(t), t \geq 0\}$ is stochastic process. Here, the time $t$ is continuous, the state space of $N(t)$ is discrete and integral valued and the process is integral valued.

Under certain conditions, the number of telephone calls, arrivals of customers for service at a counter, number of accident at a certain place, etc., follows Poisson process.

**Postulates for Poisson Process**

(i) **Independence:** $N(t)$ is independent of the number of occurrences of the event E is an interval prior to the interval $(0, t)$.

(ii) **Homogeneity in time:** $p_n(t)$ depends only on the length t of the interval and is independent of where this interval is situated.

(iii) **Regularity:** In an interval of infinitesimal length has $h$, the probability of exactly one occurrence is $\lambda h + o(h)$ and that of more than one occurrence is of $o(h)$, where $o(h)$ means, $\dfrac{o(h)}{\lambda} \to 0$ as $h \to 0$.

In other words, if the interval $(t, t+h)$ is of short duration $h$, then

1. probability of one occurrence $p_1(h) = \lambda h + o(h)$,
2. probability of $k(= 2, 3, \ldots)$ occurrences is of $o(h)$, i.e., $\sum\limits_{k=2}^{\infty} p_k(h) = o(h)$
3. Since $\sum\limits_{k=0}^{\infty} p_k(h) = 1$, therefore $p_0(h) = 1 - \sum\limits_{k=1}^{\infty} p_k(h) = 1 - \lambda h + o(h)$.

**Theorem 2.1** *Under the above postulates, $N(t)$ follows Poisson distribution with mean $\lambda t$, i.e. $p_n(t)$ is given by the following Poisson law*

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \; n = 0, 1, 2 \ldots \tag{2.3}$$

**Proof.** Let $p_n(t+h)$ be the probability of occurrence of $n(\geq 1)$ events in the time interval $(0, t+h)$.

Therefore,

$$\begin{aligned} p_n(t+h) &= P(n \text{ events occur in (0,t) and no event occur in (t,t+h)}) \\ &\quad + P((n-1) \text{ events occur in (0,t) and 1 event occurs in (t,t+h)}) \\ &\quad + P((n-2) \text{ events occur in (0,t) and 2 events occur in (t,t+h)}) \\ &\quad + \ldots \\ &\quad + P(\text{ no events occur in (0,t) and n events occur in (t,t+h)}) \\ &= \{p_n(t)(1-\lambda h) + o(h)\} + \{p_{n-1}(t)(\lambda h) + o(h)\} + \{o(h) + \ldots + o(h)\} \\ &= p_n(t)(1-\lambda h) + p_{n-1}(t)(\lambda h) + o(h),\ n \geq 1 \\ \text{or, } \frac{p_n(t+h) - p_n(t)}{h} &= -p_n(t) + \lambda p_{n-1}(t) + \frac{o(h)}{h} \end{aligned}$$

Taking $h \to 0$,

$$p'_n(t) = -\lambda p_n(t) + \lambda p_{n-1}(t),\ n \geq 0 \tag{2.4}$$

For $n = 0$,

$$\begin{aligned} p_0(t+h) &= P(\text{ no events occur in (0,t) and no events occur in (t,t+h)}) \\ &= p_0(t)(1-\lambda h) + o(h) \\ \text{or, } \frac{p_0(t+h) - p_0(t)}{h} &= -\lambda p_0(t) + \frac{O(h)}{h} \end{aligned}$$

As $h \to 0$,

$$p'_0(t) = -\lambda p_0(t) \tag{2.5}$$

**Initial conditions**

$p_0(0) = 1$ and $p_n(0) = 0$, for $n \geq 1$. From (2.5), $\frac{p'_0(t)}{p_0(t)} = \lambda$. Integrating, we get $\log p_0(t) = -\lambda t + \log C_1$ or, $p_0(t) = C_1 e^{-\lambda t}$. As $p_0(0) = 1, C_1 = 1$ therefore, $p_0(t) = e^{-\lambda t}$.

From (2.4), we have for n=1,

$p'_1(t) = -\lambda p_1(t) + \lambda p_0(t)$,

i.e., $p'_1(t) + \lambda p_1(t) = \lambda e^{-\lambda t}$.

This is a linear equation in $p_1(t)$ and integrating we get

$p_1(t)e^{\lambda t} = \lambda t + C_2$.

As $p_1(0) = 0, C_2 = 0$ and hence, $p_1(t) = \frac{\lambda t e^{-\lambda t}}{1!}$.

Let $p_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{m!}$ be the solution of (2.4). Therefore, for $n = m+1$, (2.4) becomes

$$\begin{aligned} p'_{m+1}(t) + \lambda p_{m+1}(t) &= \lambda p_m(t) \\ &= \frac{\lambda(\lambda t)^m e^{-\lambda t}}{m!} \end{aligned}$$

This is also a linear equation in $p_{m+1}(t)$ and multiplying the above equation by $e^{\lambda t}$, we get

$$\frac{d}{dt}\big(p_{m+1}(t)e^{\lambda t}\big) = \frac{\lambda(\lambda t)^m}{m!}.$$

Integrating, we get

$$p_{m+1}(t)e^{\lambda t} = \frac{(\lambda t)^{m+1}}{(m+1)!} + C_3$$

$$\text{or,}\quad p_{m+1}(t) = \left(\frac{(\lambda t)^{m+1}}{(m+1)!} + C_3\right)e^{-\lambda t}$$

As $p_{m+1}(0)$ =0 we have $C_3 = 0$, therefore, we get

$$p_{m+1}(t) = \frac{(\lambda t)^{m+1}}{(m+1)!}e^{-\lambda t}.$$

Hence

$$p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{(n)!},\ n = 0, 1, 2, \ldots$$

**Note 2.2.1** *The mean and variance of Poisson process with parameter $\lambda t$ are $\lambda t$. The mean number of occurrences per unit time ($t = 1$), i.e., in an interval of unit length is $\lambda$. The mean rate $\lambda$ per unit time is known as the parameter of the Poisson process.*

*The mean and variance of $N(t)$ are functions of $t$, in fact, its distribution is dependent on $t$.*

*Postulate 1 implies that Poisson process is Markovian; Postulate 2 that Poisson process is time homogeneous; Postulate 3 that in an infinitesimal interval of length $h$, the probability of exactly one occurrence is approximately proportional to the length $h$ of that interval and that of the simultaneous occurrence of two or more events is extremely small.*

*Poisson process has independent as well as stationery increments. Again for every $t$, future increments of a Poisson process are independent of the process generated.*

### 2.2.1 Properties of Poisson Process

**Property 2.2.1 (Additive Property)** *Some of two independent Poisson processes is a Poisson process.*

**Proof.** Let $N_1(t)$ and $N_2(t)$ be two poisson processes with parameters $\lambda_1, \lambda_2$ respectively and let $N(t) = N_1(t) + N_2(t)$.
The probability generating function (p.g.f) of $N_i(t), i = 1, 2$ is

$$E(S^{N_i(t)}) = e^{\lambda_i(s-1)t}.$$

The p.g.f of $N(t)$ is

$$\begin{aligned} E\big(S^{N_(t)}\big) &= E\big(S^{N_1(t)+N_2(t)}\big) \\ &= E\big(S^{N_1(t)}\big)E\big(S^{N_2(t)}\big) \text{ Since } N_1(t) \text{ and } N_2(t) \text{ are independent} \end{aligned}$$

Now,

$$E(S^{N(t)}) = e^{\lambda_1(s-1)t}e^{\lambda_2(s-1)t} = e^{(\lambda_1+\lambda_2)(s-1)t}.$$

Thus $N(t)$ is a Poisson process with parameter $\lambda_1 + \lambda_2$.

**Property 2.2.2** *If $\{N(t)\}$ is a Poisson process then the correlation (auto-co-relation) coefficient between $N(t)$ and $N(t+s)$ is $\left\{\frac{t}{t+s}\right\}^{\frac{1}{2}}$.*

*Proof.* Let $\lambda$ be the parameter of the Poisson process, then

$$\text{mean } = E(N(T) = \lambda T,\ Var(N(T)) = \lambda T,$$
$$E(N^2(T)) = \lambda T + (\lambda T)^2 \text{ for } T = t \text{ and } t+s.$$

Since $N(t)$ and $\{N(t+s) - N(t)\}$ are independent, $\{N(t),\ t \geq 0\}$ being a Poisson process.

$$\begin{aligned} E\{N(t)N(t+s)\} &= E\left[N(t)\{N(t+s) - N(t) + N(t)\}\right] \\ &= E\{N^2(T)\} + E\{N(t)\}E\{N(t+s) - N(t)\} \end{aligned}$$

Hence

$$E\{N(t)N(t+s)\} = (\lambda t + \lambda^2 t^2) + \lambda t \cdot \lambda s.$$

Thus the autocovariance between $N(t)$ and $N(t+s)$ is given by

$$\begin{aligned} Cov(t, t+s) &= E\{N(t)N(t+s)\} - E\{N(t)\}E\{N(t+s)\} \\ &= (\lambda t + \lambda^2 t^2 + \lambda^2 ts) - \lambda t(\lambda t + \lambda s) = \lambda t. \end{aligned}$$

Hence the aotocorrelation is

$$\begin{aligned} \rho(t, t+s) &= \frac{Cov(t, t+s)}{\sqrt{VarN(t)}\sqrt{VarN(t+s)}} \\ &= \sqrt{\left(\frac{t}{t+s}\right)}. \end{aligned}$$

## 2.3 Pure Birth Process

Let $\lambda$ be the rate of birth in a population. Here the probability that $k$ events occur in the interval $(t, t+h)$ given that $n$ events occured in the interval $(0, t)$ is given by

$$p_k(h) = \begin{cases} \lambda h + O(h), & k = 1 \\ O(h), & k \geq 2 \\ 1 - \lambda h + O(h), & k = 0. \end{cases}$$

$p_k(h)$ is independent of $n$ as well as t. We generalize the process by considering that $\lambda$ is not a constant but is a function of n or t or both; the resulting process will still be Markovian in nature.

Now, we consider the $\lambda$ is a function of n, the population size. We assume that

$$p_k(h) = \begin{cases} \lambda_n h + O(h), & k = 1 \\ O(h), & k \geq 2 \\ 1 - \lambda_n h + O(h), & k = 0. \end{cases}$$

Now,

$$\begin{aligned} p_n(t+h) &= \text{probability of having n persons in the system in an interval (0,t+h)} \\ &= \text{probability of having n persons in (0,t)} \times \text{ probability of no birth in (t,t+h)} \\ &+ \text{probability of having (n-1) persons in (0,t)} \times \text{ probability of one birth in (t,t+h)} \\ &+ \text{probability of having (n-i) persons in (0,t)} \times \text{ probability of one birth in (t,t+h)}, i \geq 2 \\ &= p_n(t)(1 - \lambda_n h) + p_{n-1}(t)\lambda_{n-1}h + O(h),\ n \geq 1. \\ \text{or, } \frac{p_n(t+h) - p_n(t)}{h} &= -\lambda_n p_n(t) + p_{n-1}(t)\lambda_{n-1} + \frac{O(h)}{h}. \end{aligned}$$

As $h \to 0$

$$p_n^{'}(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_n(t),\ n \geq 1.$$

For $n = 0$,

$$\begin{aligned} p_0(t+h) &= \text{probability of having no persons in (0,t)} \times \text{ probability of no birth in (t,t+h)} \\ &= p_0 t(1 - \lambda_0 h) + O(h) \\ \text{or, } \frac{p_0(t+h) - p_0(t)}{h} &= -\lambda_0 p_0(t) + \frac{O(h)}{h}. \end{aligned}$$

As $h \to 0$

$$p_0^{'}(t) = -\lambda_0 p_0(t). \tag{2.6}$$

In particular, if $\lambda_n = n\lambda$ then this pure birth process is called Yule-Furry process.

In this case,

$$\begin{aligned} p_n^{'}(t) &= -n\lambda p_n(t) + (n-1)\lambda p_n(t),\ n \geq 1 \qquad (2.7) \\ \text{and } \quad p_0^{'}(t) &= 0. \end{aligned}$$

Let the initial conditions be $p_1(0) = 1, p_i(0) = 0$ for $i \neq 1$, the process started with only one member at time $t = 0$.

For $n = 1$,

$$p_1'(t) = -\lambda p_1(t)$$
$$\text{or,} \quad \frac{dp_1(t)}{p_1(t)} = -\lambda dt$$

Integrating,

$$\log p_1(t) = -\lambda t + \log c_1$$
$$\text{or, } p_1(t) = c_1 e^{-\lambda t}.$$

Putting $p_1(0) = 1$ we have $c_1 = 1$.

$$\therefore p_1(t) = e^{-\lambda t}.$$

For $n = 2$, we have

$$p_2'(t) = -2\lambda p_2(t) + \lambda p_1(t)$$
$$\text{or, } p_2'(t) + 2\lambda p_2(t) = \lambda p_1(t) = \lambda e^{-\lambda t}.$$

This is a linear equation of $p_2(t)$ and its I.F $= e^{2\lambda t}$.

Multiplying above equation by $e^{2\lambda t}$, we get

$$\frac{d}{dt}\left(p_2(t)e^{2\lambda t}\right) = \lambda e^{\lambda t}.$$

Integrating we obtain

$$p_2(t)e^{2\lambda t} = \int \lambda e^{\lambda t} dt = e^{\lambda t} + c_2.$$

Since $p_2(0) = 0$ therefore, $c_2 = -1$.
Thus, $p_2(t) = e^{-2\lambda t}(e^{\lambda t} - 1) = e^{-\lambda t}(1 - e^{-\lambda t})$.
Therefore, $p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$ holds for $n = 1, 2$. We assume that the above relation is true for $n = m$. Then $p_m(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{m-1}$.
From (2.7), for $n = m + 1$,

$$p_{m+1}'(t) = -(m+1)\lambda p_{m+1}(t) + m\lambda p_m(t)$$
$$\text{or,} \quad p_{m+1}'(t) + (m+1)\lambda p_{m+1}(t) = m\lambda p_m(t) = m\lambda e^{\lambda t}(1 - e^{-\lambda t})^{m-1}$$

which is also a linear equation of $p_{m+1}(t)$ and I.F. is $e^{(m+1)\lambda t}$, we get

$$\frac{d}{dt}(p_{m+1}(t)e^{(m+1)\lambda t}) = m\lambda e^{m\lambda t}(1 - e^{-\lambda t})^{m-1}.$$

Integrating, we have

$$p_{m+1}(t)e^{(m+1)\lambda t} = \int m\lambda e^{\lambda t}(e^{\lambda t} - 1)^{m-1} dt$$

Putting $e^{\lambda t} - 1 = z$, we get

$$\begin{aligned} p_{m+1}(t)e^{(m+1)\lambda t} &= \int mz^{m-1}dz = z^m + c_3 \\ &= (e^{\lambda t} - 1)^m + c_3 \end{aligned}$$

Since $p_{m+1}(t) = 0,\ c_3 = 0$.
Hence, $p_{m+1}(t) = (e^{-\lambda t} - 1)^m e^{-(m+1)\lambda t} = e^{-\lambda t}(1 - e^{-\lambda t})^m$.Therefore the process, i.e. the result holds for $n = m + 1$. Hence by mathematical induction

$$p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^n,\ n \geq 1. \tag{2.8}$$

Solving, $p_0'(t) = 0$, we get $p_0(t) = 0$ [As $p_0(0) = 0$.]

**Note 2.3.1** *The probability generating function (pgf) is given by*

$$\begin{aligned} P(s,t) = \sum_{n=1}^{\infty} p_n(t)s^n &= \sum_{n=1}^{\infty} \{e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}\}s^n \\ &= e^{-\lambda t}s\sum_{n=1}^{\infty}(1 - e^{-\lambda t})^{n-1}s^{n-1} \\ &= e^{-\lambda t}s\frac{1}{1 - s(1 - e^{-\lambda t})} \\ &= \frac{se^{-\lambda t}}{1 - s(1 - e^{-\lambda t})}. \end{aligned} \tag{2.9}$$

## 2.4 Birth and Death Process

In this process the number of individuals will increase as well as decrease. The following assumptions are made to describe the birth and death process.
**Assumption:**

(i) Probability of one birth during $(t, t + h),\ (h \geq 0)$ is $\lambda_n h + O(h)$ where n is the number of items in a population at time t and $\lambda_n$ is the birth rate.

(ii) Probability of more than one birth during $(t, t+h)$ is $O(h)$. Hence the probability of no birth during $(t, t + h)$ is $1 - \lambda_n h + O(h)$.

(iii) Probability of one death during $(t, t + h)$ is $\mu_n h + O(h),\ \mu_n$ is the death rate.

(iv) Probability of more than one death during $(t, t+h)$ is $O(h)$. Hence the probability of no death during $(t, t + h)$ is $1 - \mu_n h + O(h)$.

Let $p_n(t)$ be the probability of having population size $n$ at time $t$.

$$\begin{aligned}
p_n(t+h) &= \text{probability of having population size n at time t+h} \\
&= \text{(probability of n items in (0,t)} \times \text{probability of no birth and no death during (t,t+h))} \\
&\quad + \text{(probability of (n+1) items in (0,t)} \times \text{probability of no birth and probability of} \\
&\quad \text{one death during (t,t+h))+(probability of (n-1) items in (0,t)} \times \text{probability of one} \\
&\quad \text{birth and probability of no death during (t,t+h)+(probability of n items in (0,t)}\times \\
&\quad \text{probability of one birth and one death during (t,t+h))+O(h)} \\
&= p_n(t)\{1-\lambda_n h+O(h)\}\{1-\mu_n h+O(h)\}+p_{n+1}(t)\{1-\lambda_{n+1}h+O(h)\}\{\mu_{n+1}h+O(h)\} \\
&\quad +p_{n-1}(t)\{\lambda_{n-1}h+O(h)\}\{1-\mu_{n-1}h+O(h)\}+p_n(t)\{\lambda_n h+O(h)\}\{\mu_n h+O(h)\}+O(h) \\
&= p_n(t)\{1-\lambda_n h-\mu_n h)\}+p_{n+1}(t)\{\mu_{n+1}h\}+p_{n-1}\{\lambda_{n-1}h\}+o(h) \\
&\qquad \text{[neglecting the second and higher powers of h]}
\end{aligned}$$

$$\text{or, } p_n(t+h)\ -p_n(t) = -h(\lambda_n+\mu_n)p_n(t)+\mu_{n+1}hp_{n+1}(t)+\lambda_{n-1}hp_{n-1}(t)+O(h).$$

Dividing both sides by h and taking $h \to 0$, we get

$$p_n^{'}(t) = -(\lambda_n+\mu_n)p_n(t)+\mu_{n+1}p_{n+1}(t)+\lambda_{n-1}p_{n-1}(t),\ n \geq 1. \tag{2.10}$$

For $n = 0$, we have

$$\begin{aligned}
p_0(t+h) &= p_0(t)\{1-\lambda_0 h+O(h)\}\{1-\mu_0 h+O(h)\}+p_1(t)\{1-\lambda_1 h+O(h)\}\{\mu_1 h+O(h)\} \\
&= p_0(t)-\lambda_0 hp_0(t)-\mu_0 hp_0(t)+\mu_1 hp_1(t)+O(h)
\end{aligned}$$

$$\text{or, } p_0(t+h)-p_0(t) = -\lambda_0 hp_0(t)-\mu_0 hp_0(t)+\mu_1 hp_1(t)+O(h)$$

$$\text{or, } \frac{p_0(t+h)-p_0(t)}{h} = -\lambda_0 p_0(t)-\mu_0 p_0(t)+\mu_1 p_1(t)+\frac{O(h)}{h}.$$

Taking limit $h \to 0$, we get

$$p_0^{'}(t) = -(\lambda_0+\mu_0)p_0(t)+\mu_1 p_1(t). \tag{2.11}$$

**Initial condition:** Suppose, initially there are $i$ members in the system, i.e.

$$p_n(0) = \begin{cases} 0, n \neq i \\ 1, n = i \end{cases}. \tag{2.12}$$

The equations (2.10) and (2.11) are the equations of the birth and death process with initial conditions given by (2.12).

**Condition of existence of the solution of (2.10) and (2.11)**

For arbitrary $\lambda_n \geq 0, \mu_n \geq 0$, there always exists a solution $p_n(t)(\geq 0)$ such that $\sum p_n(t) \leq 1$. If $\lambda_n$ and $\mu_n$ are bounded, the solution is unique and $\sum p_n(t) = 1$.

**Note 2.4.1** *When $\lambda_n = \lambda$ i.e. $\lambda_n$ is independent of the population size n, then the increase may be thought of as due to an external source such as immigration.*

*When $\lambda_n = n\lambda$, we have the case of linear birth, the rate of birth in unit interval being $\lambda$ per individual.*

*When $\mu_n = \mu$, decrease may be thought of as due to a factor such as emigration.*

*When $\mu_n = n\mu$, we have the case of linear death, the rate of death in unit interval being $\mu$ per individual.*

### 2.4.1 Solution of linear growth process

**(a) Generating function**

When $\lambda_n = n\lambda$ and $\mu_n = n\mu (n \geq 1)$ then the process is called linear growth process, where $\lambda_0 = 0,\ \mu_0 = 0$. If $X(t)$ denotes the total number of members at time t, then from (2.10) and (2.11), we have the following differential difference equations for $p_n(t) = P(X(t) = n)$,

$$p_n'(t) = -n(\lambda+\mu)p_n(t) + \lambda(n-1)p_{n-1}(t) + \mu(n+1)p_{n+1}(t),\ n \geq 1 \tag{2.13}$$

$$\text{and}\quad p_0'(t) = \mu p_1(t). \tag{2.14}$$

If the initial population size is $i$, i.e. $X(0) = i$, then we have the initial condition $p_i(0) = 1$ and $p_n(0) = 0,\ n \neq i$.

Let $P(s,t) = \sum\limits_{n=1}^{\infty} p_n(t)s^n$ be the probability generating function of $\{p_n(t)\}$.

Then $\dfrac{\partial P}{\partial s} = \sum\limits_{n=1}^{\infty} np_n(t)s^{n-1}$ and $\dfrac{\partial P}{\partial t} = \sum\limits_{n=1}^{\infty} p_n'(t)s^n$.

Multiplying (2.13) by $s^n$ and adding for $n = 1, 2, 3 \ldots$ and adding (2.14) with it, we get

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -(\lambda+\mu)\sum_{n=1}^{\infty} np_n(t)s^n + \lambda\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)s^n \\
&\quad + \mu\left\{\sum_{n=1}^{\infty}(n+1)p_{n+1}(t)s^n + p_1(t)\right\} \\
&= -(\lambda+\mu)s\sum_{n=1}^{\infty} np_n(t)s^{n-1} + \lambda s^2\sum_{n=1}^{\infty} np_n(t)s^{n-1} + \mu\sum_{n=1}^{\infty} np_n(t)s^{n-1} \\
&= -(\lambda+\mu)s\frac{\partial P}{\partial s} + \lambda s^2\frac{\partial P}{\partial s} + \mu\frac{\partial P}{\partial s} \\
&= \left\{\mu - (\lambda+\mu)s + \lambda s^2\right\}\frac{\partial P}{\partial s}.
\end{aligned}$$

This is a partial differential equation of Lagrangian type. The initial condition is $X(0) = i$.

The Lagrange's equation becomes

$$\frac{dt}{1} = \frac{ds}{-(s-1)(\lambda s-\mu)} = \frac{dp}{0}$$

$$\text{or,}\quad dt = \frac{ds}{-(s-1)(\lambda s-\mu)} = \frac{1}{\lambda-\mu}\left[\frac{1}{1-s}+\frac{\lambda}{\lambda s-\mu}\right]ds.$$

Integrating we get

$$(\lambda-\mu)t = \log[(\lambda s-\mu)/(1-s)] + \log c$$

$$\text{or,}\quad \frac{1-s}{\lambda s-\mu}e^{(\lambda-\mu)t} = c. \tag{2.15}$$

$$\text{Again,}\quad \frac{dt}{1} = \frac{dP}{0} \text{ gives } P = \text{ constant} = g(s)$$

$$\text{or, } P(s,t) = g(s) \tag{2.16}$$

That is, $\sum_{n=0}^{\infty} p_n(t)s^n = g(s)$ or, $s^i = g(s)$ [$\because p_n(0) = 0$ and $n \neq i, p_i(0) = 1.$]
Hence (2.16) becomes

$$P(s,t) = s^i. \tag{2.17}$$

When $t = 0$ equation (2.15) becomes

$$\frac{1-s}{\lambda s-\mu} = c, \quad \text{or, } s = \frac{1+\mu c}{1+\lambda c}.$$

Therefore, (2.17) becomes

$$P(s,t) = \left(\frac{1+\mu c}{1+\lambda c}\right)^i. \tag{2.18}$$

$$\text{where } c = \frac{1-s}{\lambda s-\mu}e^{(\lambda-\mu)t} \tag{2.19}$$

Therefore,

$$P(s,t) = \left[\frac{\mu(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}{\lambda(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}\right]^i$$

$$= \left[\frac{\mu\{1-e^{-(\lambda-\mu)t}\}-\{\mu-\lambda e^{-(\lambda-\mu)t}\}s}{\{\lambda-\mu e^{-(\lambda-\mu)t}\}-\lambda\{1-e^{-(\lambda-\mu)t}\}s}\right]^i. \tag{2.20}$$

Explicit expression for $p_n(t)$ can be obtained from the above by expanding $P(s,t)$ as a power series in $s$ for $i = 1$ as

$$p_n(t) = \{1-\alpha(t)\}\{1-\beta(t)\}\{\beta(t)\}^{n-1},\ n = 1, 2, \ldots$$

$$p_0(t) = \alpha(t),$$

where $\alpha(t) = \dfrac{\mu\{e^{(\lambda-\mu)t} - \}}{\lambda e^{(\lambda-\mu)t} - \mu}$ and $\beta(t) = \dfrac{\lambda\{e^{(\lambda-\mu)t} - 1\}}{\lambda e^{(\lambda-\mu)t} - \mu}$.

**(b) Mean population size**

The mean population size can be obtained from $P(s,t)$ by differentiating it w.r.t s and substituting $s = 1$, i.e.,

$$E\{X(t)\} = \frac{\partial P}{\partial s}\Big)_{s=1} = \sum_{n=1}^{\infty} np_n(t) = \alpha_1(t), \quad \text{(say)} .$$

Similarly,

$$E\{X^2(t)\} = \sum_{n=1}^{\infty} n^2 p_n(t) = \alpha_2(t)$$

Multiplying (2.13) by n and adding for $n = 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} np_n'(t) = -(\lambda+\mu)\sum_{n=1}^{\infty} n^2 p_n(t) + \lambda\sum_{n=1}^{\infty} n(n-1)p_{n-1}(t) + \mu\sum_{n=1}^{\infty} n(n+1)p_{n+1}(t). \tag{2.21}$$

Now,

$$\begin{aligned}\sum_{n=1}^{\infty} n(n-1)p_{n-1}(t) &= \sum_{n=1}^{\infty}\Big[(n-1)^2 + (n-1)\Big]p_{n-1}(t)\\ &= \sum_{n=1}^{\infty}(n-1)^2 p_{n-1}(t) + \sum_{n=1}^{\infty}(n-1)p_{n-1}(t)\\ &= \alpha_2(t) + \alpha_1(t).\end{aligned}$$

Again,

$$\begin{aligned}\sum_{n=1}^{\infty} n(n+1)p_{n+1}(t) &= \sum_{n=1}^{\infty}\Big[(n+1)^2 - (n+1)\Big]p_{n+1}(t)\\ &= \sum_{n=1}^{\infty}(n+1)^2 p_{n+1}(t) - \sum_{n=1}^{\infty}(n+1)p_{n+1}(t)\\ &= \{\alpha_2(t) - p_1(t)\} - \{\alpha_1(t) - p_1 t\}\\ &= \alpha_2(t) - \alpha_1(t)\\ \text{and } \sum_{n=1}^{\infty} np_n'(t) &= \alpha_1'(t).\end{aligned}$$

Hence (2.21) becomes

$$\begin{aligned}\alpha_1'(t) &= -(\lambda+\mu)\alpha_2(t) + \lambda\{\alpha_2(t) + \alpha_1(t)\} + \mu\{\alpha_2(t) - \alpha_1(t)\}\\ &= (\lambda-\mu)\alpha_1(t).\end{aligned}$$

Integrating, we get $\alpha_1(t) = ce^{(\lambda-\mu)t}$.
The initial condition gives $\alpha_0(t) = \sum_{n=1}^{\infty} np_n(0) = i$
That is $c = \alpha_1(0) = i$.
Thus

$$\alpha_1(t) = ie^{(\lambda-\mu)t} \tag{2.22}$$

Hence the mean population size when initial population size i is $ie^{(\lambda-\mu)t}$.

**Limiting case:**
As $t \to \infty$, the mean population size $\alpha_1(t) \to 0$ for $\lambda < \mu$ or to $\infty$ for $\lambda > \mu$ and to the constant value when $\lambda = \mu$.

**(c) Extinction probability**
Since $\lambda_0 = 0$, 0 is an absorbing state i.e., once the population size reaches 0, it remains at 0 thereafter. This is the interesting case of extinction of the population.

Suppose the process starts with only one member at time 0, i.e., $X(0) = 1$.
Then from (2.20)

$$P(s,t) = \frac{a-bs}{c-ds} = \frac{a}{c}\frac{1-\frac{bs}{a}}{1-\frac{ds}{c}},$$

$$\begin{aligned} \text{where} \quad & a = \mu\{1-e^{-(\lambda-\mu)t}\} \\ & b = \mu - \lambda e^{-(\lambda-\mu)t} \\ & c = \lambda - \mu e^{-(\lambda-\mu)t} \\ & d = \lambda\{1-e^{-(\lambda-\mu)t}\}. \end{aligned}$$

$$\begin{aligned} \therefore P(s,t) &= \frac{a}{c}\left(1-\frac{bs}{a}\right)\left(1-\frac{ds}{c}\right)^{-1} \\ &= \frac{a}{c}\left(1-\frac{bs}{a}\right)\left\{1+\frac{ds}{c}+\left(\frac{ds}{c}\right)^2+\left(\frac{ds}{c}\right)^3+\cdots+\left(\frac{ds}{c}\right)^{n-1}+\left(\frac{ds}{c}\right)^n+\cdots\right\}. \end{aligned}$$

Coefficient of $s^n$ in $P(s,t)$ is

$$\frac{a}{c}\left\{\left(\frac{d}{c}\right)^n - \frac{b}{a}\left(\frac{d}{c}\right)^{n-1}\right\} = p_n(t) \tag{2.23}$$

$$\text{and} \quad \frac{a}{c} = p_0(t).$$

Therefore,

$$\lim_{t\to\infty} p_0(t) = \lim_{t\to\infty} \frac{\mu\{1-e^{-(\lambda-\mu)t}\}}{\lambda-\mu e^{-(\lambda-\mu)t}}$$

$$= \begin{cases} \frac{\mu}{\lambda}, \ \lambda > \mu \\ 1, \ \lambda \le \mu \end{cases} \tag{2.24}$$

$$\text{and} \ \lim_{t\to\infty} p_n(t) = 0 \quad \text{for } n \neq 0. \tag{2.25}$$

In other words, the probability of ultimate extinction is 1 when $\mu > \lambda$ and is $\frac{\mu}{\lambda}(< 1)$ when $\mu < \lambda$.

If initially $X(0) = i$ then the probability of ultimate extinction is $\left(\frac{\mu}{\lambda}\right)^i$ for $\mu < \lambda$.

## 2.5 Markov Processes with Continuous State Space: Wiener Process

Consider that a (Brownian) particle performs a random walk such that in a small interval of time of duration $\Delta t$, the displacement of the particle to the right or to the left is also a small magnitude $\Delta x$, the total displacement $X(t)$ of the particle in time t being x. Suppose that the random variable $Z_i$ denotes the length of the ith step taken by the particle in a small interval of time $\Delta t$ and that

$$P(Z_i = \Delta x) = p \text{ and } P(Z_i = -\Delta x) = q,\ p + q = 1,\ 0 < p < 1$$

where p is independent of x and t.

Suppose that the interval of length t is divided into n equal subintervals of length $\Delta t$ and that the displacement $Z_i, i = 1, 2, \ldots, n$ in the n steps are mutually independent random variables. Then $n \times \Delta = t$ and the total displacement $X(t)$ is the sum of n i.i.d( independent identically distributed) random variables $Z_i$, i.e.,

$$X(t) = \sum_{i=1}^{n(t)} Z_i,\ n \equiv n(t) = t/\Delta t.$$

We have

$$\begin{aligned}
E(Z_i) &= z_i P(Z_i = \Delta x) + z_i P(Z_i = -\Delta x) \\
&= \Delta x p - \Delta x q = (p - q)\Delta x. \\
E(Z_i^2) &= (\Delta x)^2 p + (\Delta x)^2 q = (\Delta x)^2. \\
\therefore Var(Z_i) = E(Z_i^2) - \{E(Z_i)\}^2 &= (\Delta x)^2 - (p - q)^2 (\Delta x)^2 \\
&= 4pq(\Delta x)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
E\{X(t)\} &= nE(Z_i) = t(p - q)(\Delta x)^2/\Delta t,\ [\because n = t/\Delta t] \quad (2.26) \\
\text{and } Var(X(t)) &= nVar(Z_i) = 4pqt(\Delta x)^2/\Delta t.
\end{aligned}$$

To get a meaningful result, as $\Delta x \to 0, \Delta t \to 0$. We must have

$$\begin{aligned}
&\frac{(\Delta x)^2}{\Delta t} \to \text{ a limit} \\
&(p - q) \to 0 \text{ a multiple of } \Delta x \quad (2.27)
\end{aligned}$$

We assume that, in an interval of length $t, X(t)$ has mean value function equal to $\mu t$ and variance function equal to $t\sigma^2$, that is, we suppose that as $\Delta x \to 0, \Delta t \to 0$ in such a way that (2.27) are satisfied and

$$E\{X(t)\} = \mu t \text{ and } Var\{X(t)\} = t\sigma^2. \tag{2.28}$$

Therefore from (2.26), we have

$$\frac{(p-q)\Delta x}{\Delta t} \to \mu, \ \frac{4pq(\Delta x)^2}{\Delta t} \to \sigma^2. \tag{2.29}$$

The relation (2.27) and (2.29) will be satisfied when

$$\Delta x = \sigma\sqrt{\Delta t} \tag{2.30}$$

$$\text{and } p = \frac{1}{2}\{1 + \mu\sqrt{\Delta t}/\sigma\}, \quad q = \frac{1}{2}\{1 - \mu\sqrt{\Delta t}/\sigma\} \tag{2.31}$$

Now, since $Z_i$ are i.i.d random variables, the sum $\sum\limits_{i=1}^{n(t)} Z_i = X(t)$ for large $n(t)(\equiv n)$ is asymptotically normal with mean $\mu t$ and variance $t\sigma^2$(by central limit theorem for equal components).

Here t represents the length of the interval of time during which the displacement, that takes place is equal to the increment $X(t) - X(0)$. We thus find that for $0 < s < t, \{X(t) - X(s)\}$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$. Further the increments $\{X(t) - X(0)\}$ and $\{X(t) - X(s)\}$ are mutually independent this implies that $\{X(t)\}$ is Markov process.

We now define a Wiener process or a Brownian process as follows:
The stochastic process $\{X(t), t \geq 0\}$ is called a Wiener process (or a Wiener Einstein process ar a Brownian motion process) with drift $\mu$ and variance parameter $\sigma^2$ if

1. $X(t)$ has independent increments, i.e., for every pair of disjoint intervals of time $(s,t)$ and $(u,v)$, where $s \leq t \leq u \leq v$, the random variables $\{X(t) - X(s)\}$ and $\{X(v) - X(u)\}$ are independent.

2. Every increment $\{X(t) - X(s)\}$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

Since $X(t) - X(0)$ is normally distributed with mean $\mu t$ and variance $t\sigma^2$, the transition p.d.f. p of a Wiener process is given by

$$\begin{aligned} p(x_0, x, t)dx &= P\{x \leq X(t) < x + dx / X(0) = x_0\} \\ &= \frac{1}{\sigma\sqrt{2\pi t}} e^{-(x-x_0-\mu t)^2/(2\sigma^2 t)} dx, \quad -\infty < x < \infty \end{aligned} \tag{2.32}$$

A Wiener process $\{X(t), t \geq 0\}$ with $X(0) = 0, \mu = 0, \sigma = 1$ is called a standard Wiener process.

### 2.5.1 Differential equations for a Wiener process

Let $\{X(t),\ t \geq 0\}$ be a Wiener process. We can consider the displacement in such a process as being caused by the motion of a particle undergone displacements of small magnitude in a small interval of time. Suppose that $(t - \Delta t, t)$ is an infinitesimal interval of length $\Delta t$ and the particle makes in this interval a shift equal to $\Delta x$ with probability p or a shift equal to $-\Delta x$ with probability $q = 1 - p$. Suppose that p and q are independent of x and t. Let the transition probability that the particle has a displacement from x to $x + \Delta x$ in the interval (0,t), given that it started from $x_0$ at time 0, be $p(x_0, x; t)\Delta x$.

Therefore by Taylor's series

$$p(x_0, x \pm \Delta x; t - \Delta t) = p(x_0, x; t) - \Delta t\frac{\partial p}{\partial t} \pm \Delta x\frac{\partial p}{\partial x} + \frac{1}{2}(\pm\Delta x)^2\frac{\partial^2 p}{\partial^2 x} + O(\Delta t). \tag{2.33}$$

For simple probability arguments we have

$$p(x_0, x; t)\Delta x = p \cdot p(x_0, x - \Delta x; t - \Delta t)\Delta x + q \cdot p(x_0, x + \Delta x; t - \Delta t)\Delta x$$

$$p(x_0, x; t) = p \cdot p(x_0, x - \Delta x; t - \Delta t) + q \cdot p(x_0, x + \Delta x; t - \Delta t) \tag{2.34}$$

Using (2.33) and (2.34) becomes

$$\begin{aligned} p(x_0, x; t) &= p\left\{p(x_0, x; t) - \Delta t\frac{\partial p}{\partial t} - \Delta x\frac{\partial p}{\partial x} + \frac{1}{2}(\Delta x)^2\frac{\partial^2 p}{\partial^2 x} + O(\Delta t)\right\} \\ &\quad + q\left\{p(x_0, x; t) - \Delta t\frac{\partial p}{\partial t} + \Delta x\frac{\partial p}{\partial x} + \frac{1}{2}(-\Delta x)^2\frac{\partial^2 p}{\partial^2 x} + O(\Delta t)\right\} \\ &= p(x_0, x; t) - \Delta t\frac{\partial p}{\partial t} - \Delta x(p - q)\frac{\partial p}{\partial x} + \frac{1}{2}(\Delta x)^2\frac{\partial^2 p}{\partial^2 x} + O(\Delta t) \\ \text{or, } \frac{\partial p}{\partial t} + \frac{\Delta x}{\Delta t}(p - q)\frac{\partial p}{\partial x} &= \frac{1}{2}\frac{(\Delta x)^2}{\Delta t}\frac{\partial^2 p}{\partial^2 x} + \frac{O(\Delta t)}{\Delta t}. \end{aligned}$$

Taking limits as $\Delta t \to 0,\ \Delta x \to 0$ we get from (2.29), (2.30) and (2.31)

$$p = \frac{1}{2},\ q = \frac{1}{2}.$$

Using these limits we get

$$\frac{\partial p}{\partial t} = -\mu\frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial^2 x}, p = p(x_0, x; t) \tag{2.35}$$

The equation is known as the forward diffusion equation of the Wiener process. The backward diffusion equation of the process in the form

$$\frac{\partial p}{\partial t} = \mu\frac{\partial p}{\partial x_0} + \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial^2 x_0}, p = p(x_0, x; t) \tag{2.36}$$

The solution of (2.35) and (2.36) yield $p(x_0, x, t)$ as a normal density of the form given in (2.32). The equation for a Wiener process with drift $\mu = 0$ is

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial^2 x} \tag{2.37}$$

which is known as the heat equation.

## 2.6 Branching Process

WE consider first the discrete time space. Suppose that we start with an initial set of objects or individuals which form the 0th generation-these objects are called ancestors. The offsprings produced or the objects generated by the objects of the 0th generation are the direct descendants of the ancestors, and are said to form the first generation; the objects generated by these of the first generation or the direct descendants of the first generation from the second generation and so on, the direct descendants of the rth generation form the (r+1)th generation. The number of objects of the rth generation (r=0,1,2,. . . ) is a random variable. We assume that the objects reproduce independently of other objects, i.e. there is no interference.

Let the random variables $X_0, X_1, X_2, \ldots$ denote the sizes of (or the numbers of objects in) the 0th, 1st, 2nd, . . . generations respectively. Let the probability that an object (irrespective of the generation to which belongs) generates k similar objects be denoted by $p_k$, where $p_k \geq 0, k = 1, 2, \ldots$ and $\sum_k p_k = 1$.

The sequence $\{X_n,\ n = 0, 1, 2 \ldots\}$ constitutes a Galton-Watson branching process (or simply a G.W. branching process) with offspring distribution $\{p_k\}$. The process is also called Bienayame-Galton-Watson process.

### 2.6.1 Properties of generating functions of branching processes

A Galton-Watson process is a Markov chain $\{X_n,\ n = 0, 1, 2, \ldots\}$ having state space N (set of non-negative integers), such that

$$X_{n+1} = \sum_{r=1}^{X_n} Z_r, \tag{2.38}$$

where $Z_r$ is independently identically distributed random variable with distribution $\{p_k\}$. Let

$$P(s) = \sum_k P(Z_r = k)s^k = \sum_k p_k s^k \tag{2.39}$$

be the p.g.f. of $\{Z_r\}$ and let

$$P_n(s) = \sum_k P(X_n = k)s^k,\ n = 0, 1, 2, \ldots \tag{2.40}$$

be the p.g.f of $\{X_n\}$.
We assume that $X_0 = 1$; clearly $P_0(s) = s$ and $P_1(s) = P(s)$. The random variables $X_1$ and $Z_r$(for any r) both give the same offspring distribution.

**Theorem 2.2** *The generating function $P_n(s)$ satisfy the following relations:*

$$P_n(s) = P_{n-1}(p(s)) \tag{2.41}$$

$$\text{and} \quad P_n(s) = P(P_{n-1}(s)). \tag{2.42}$$

*Proof.* We have, for n=1,2,. . .

$$\begin{aligned} P(X_n = k) &= \sum_{j=0}^{\infty} P(X_n = k/X_{n-1} = j)P(X_{n-1} = j) \\ &= \sum_{j=0}^{\infty} P\left(\sum_{r=1}^{j} Z_r = k\right) P(X_{n-1} = j) \end{aligned}$$

$$\begin{aligned} \therefore P_n(s) &= \sum_{k=0}^{\infty} P(X_n = k)s^k \\ &= \sum_{k=0}^{\infty} s^k \left[\sum_{j=0}^{\infty} P\left(\sum_{r=1}^{j} Z_r = k\right) P(X_{n-1} = j)\right] \\ &= \sum_{k=0}^{\infty} P(X_{n-1} = j) \left[\sum_{k=0}^{\infty} P\left(\sum_{r=1}^{j} Z_1 + Z_2 + \cdots + Z_j = k\right) s^k\right]. \end{aligned} \tag{2.43}$$

The expression within the square bracket being the p.g.f of the sum $Z_1 + Z_2 + \cdots + Z_j$ of j i.i.d random variables each with p.g.f. P(s), equals $[P(s)]^j$. Thus

$$\begin{aligned} P_n(s) &= \sum_{j=0}^{\infty} P(X_{n-1} = j)[P(s)]^j \\ \text{or,} \quad P_n(s) &= P_{n-1}(P(s)). \end{aligned}$$

Putting $n = 2, 3, 4 \dots$ we get (when $X_0 = 1, P_1 = P$)

$$P_2(s) = P_1(P(s)) = P(P(s)),\ P_3(s) = P_2(P(s)),\ P_4(s) = P_3(P(s)),\ \text{and so on.}$$

$$\text{Therefore,}\ \ P_n(s) = P_{n-1}(P(s)) = P_{n-1}(P(P(s))) = P_{n-2}(P_2(s)). \tag{2.44}$$

For n=3, $P_3(s) = P_1(P_2(s)) = P(P_2(s))$.
Again, $P_n(s) = P_{n-3}(P(P_2(s))) = P_{n-3}(P_3(s))$ and for n=4,
$P_4(s) = P_1(P_3(s)) = P(P_3(s))$.
Thus $P_n(s) = P_{n-k}(P_k(s)),\ k = 0, 1, 2, \dots, n$ and for $k = n - 1$
$P_n(s) = P_1(P_{n-1}(s)) = P(P_{n-1}(s))$.

**Note 2.6.1** *When $X_0 = i \neq 1$ then (2.42) holds but (2.42) does not hold.*

$$p'(1) = E(Z_r) = E(X_1) = m.$$

**Theorem 2.3** *If $m = E(X_1) = \sum_{k=0}^{\infty} kp_k$ and $\sigma^2 = Var(X_1)$ then*

$$E(X_n) = m^n \text{ and } Var(X_n) = \begin{cases} \frac{m^{n-1}(m^n-1)}{m-1}\sigma^2, & \text{if } m \neq 1 \\ n\sigma^2, & \text{if } m = 1. \end{cases}$$

*Proof.* We have from (2.41),

$$P_n(s) = P_{n-1}(P(s)). \tag{2.45}$$

Therefore, $P_n^{'}(s) = P_{n-1}^{'}(P(s))P^{'}(s)$
Hence $P_n^{'}(1) = P_{n-1}^{'}(1)P^{'}(1)$ [$\because$ $P(s) = 1$]
$= mP_{n-1}^{'}(1)$.
Therefore, $P_n^{'}(1) = m^2 P_{n-2}^{'}(1) = m^3 P_{n-3}^{'}(1) = \ldots = m^{n-1}P^{'}(1) = m^n$.
Thus $E(X_n) = P_n^{'}(1) = m^n$.
Again differentiating (2.45) we get $P_n^{''}(s) = P_{n-1}^{''}(P(s))\{P^{'}(s)\}^2 + P_{n-1}^{'}(P(s))P^{''}(s)$.
At $s = 1$,

$$\begin{aligned}
P_n^{''}(1) &= P_{n-1}^{''}(P(1))\{P^{'}(1)\}^2 + P_{n-1}^{'}(P(1))P^{''}(1) \\
&= P_{n-1}^{''}(1)m^2 + P_{n-1}^{'}(1)A, \quad \text{where } A = P^{''}(1) \\
&\qquad\qquad [\because P^{'}(1) = m \text{ and } P(1) = 1] \\
&= P_{n-1}^{''}(1)m^2 + m^{n-1}A \\
&= (P_{n-2}^{''}(1)m^2 + m^{n-2}A)m^2 + m^{n-1}A \\
&= P_{n-2}^{''}(1)m^4 + m^nA + m^{n-1}A \\
&= (P_{n-3}^{''}(1)m^2 + m^{n-3}A)m^4 + m^nA + m^{n-1}A \\
&= P_{n-3}^{''}(1)m^6 + m^{n+1}A + m^nA + m^{n-1}A \\
&= \ldots\ldots \\
&= P_0^{''}(1)m^{2n} + (m^{2n-2} + m^{2n-1} + \ldots + m^n + m^{n-1})A \\
&= 0 + Am^{n-1}(1 + m + m^2 + \ldots + m^{n-1}) \\
&\qquad\qquad [\because P_0(s) = s, P_0^{''}(s) = 0] \\
&= Am^{n-1}\frac{m^n - 1}{m - 1} \text{ for } m \neq 1.
\end{aligned}$$

Now, $P(s) = \sum_k p_k s^k$. Therefore, $P^{'}(s) = \sum_k k p_k s^{k-1}$ and $P^{''}(s) = \sum_k k(k-1)p_k s^{k-2}$.

$$\begin{aligned}
P^{''}(1) &= \sum_k k(k-1)p_k = \sum_k k^2 p_k - \sum_k k p_k \\
&= E(Z_r^2) - E(Z_r) \\
\therefore A = P^{''}(1) &= E(Z_r^2) - \{E(Z_r)\}^2 + \{E(Z_r)\}^2 - E(Z_r) \\
&= Var(Z_r) + m^2 - m \\
&= \sigma^2 + m(m-1).
\end{aligned}$$

Similarly,

$$\begin{aligned} P_n^{''}(1) &= E(X_n^2) - \{E(X_n)\}^2 + \{E(X_n)\}^2 - E(X_n) \\ &= Var(X_n) + (m^n)^2 - m^n \end{aligned}$$

Hence

$$\begin{aligned} Var(X_n) + m^n(m^n - 1) &= \{\sigma^2 + m(m-1)\}m^{n-1}\frac{m^n - 1}{m - 1} \\ &= \sigma^2 m^{n-1}\frac{m^n - 1}{m - 1} + m.m^{n-1}(m^n - 1) \\ \text{or, } Var(X_n) &= m^{n-1}\frac{m^n - 1}{m - 1}\sigma^2 \text{ for } m \neq 1 \end{aligned}$$

When $m = 1$, then the expression of $P_n^{''}(1)$ becomes

$$P_n^{''}(1) = A(1 + 1 + 1 + \ldots + n \text{ times }) = A.n.$$

In this case, $A = \sigma^2$ and $P_n^{''}(1) = Var(X_n) + (1^n)^2 - 1^n = Var(X_n)$.
Therefore, $Var(X_n) = n\sigma^2$

Thus, $Var(X_n) = \begin{cases} \dfrac{m^{n-1}(m^n - 1)}{m - 1}\sigma^2, & \text{if } m \neq 1 \\ n\sigma^2, & \text{if } m = 1. \end{cases}$

## 2.7 Unit Summary

In this unit, Poisson process is introduced. Pure birth process and birth and death process are also studied. The probability generating function for birth and death process is determined, where birth and death rates are linear. Mean population size and extinction probability are calculated. An example of continuous time continuous state space Markov process, viz., Wiener process is presented. The differential equation of this process is also established. The Galton-Watson branching process is introduced and studied some of its properties. A list of exercises is supplied with this unit.

## 2.8 Self Assessment Questions

1. Prove that under certain conditions stated by you the number of telephone calls in a trunk line follows Poisson process. Find the mean and standard deviation of this process.

2. If $N_1(t), N_2(t)$ are two independent Poisson processes with parameters $\lambda_1, \lambda_2$ respectively, then show that $P(N_1(t) = k/N_1(t) + N_2(t) = n) = \binom{n}{k}p^k q^{n-k}$, where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$.

3. The number of accidents in a town follows a Poisson process with a mean of 2 per day and the number $X_i$ of people involved in the ith accident has the distribution $P(X_i = k) = \frac{1}{2^k},\ k \geq 1$. Find the mean and variance of the number of people involved in accidents per week.

4. Deduce the probability mass function for pure birth process. Hence deduce the probability generating function for this process.

5. State birth and death process. Find the differential difference equation for birth and death process.

6. Find the probability generating function for birth and death process when rate of birth and death are respectively $n\lambda$ and $n\mu$, where n is the population size at any time t. Assume that the initial population size is i.

7. Find the probability of ultimate extinction in the case of the linear growth process(birth and death) starting with i individuals at time 0.

8. Find the differential equation for Wiener process.

9. Show that the generating function $P_n(s)$ for branching process satisfy the following relations:

   (i) $P_n(s) = P_{n-1}(P(s))$ and

   (ii) $P_n(s) = P(P_{n-1}(s))$, where $P_1(s) = P(s)$.

10. Let $\{X_n, n \geq 0\}$ be a branching process. Show that if $m = E(X_1) = \sum_{k=0}^{\infty} kp_k$ and $\sigma^2 = Var(X_1)$ then

$$E(X_n) = m^n \text{ and } Var(X_n) = \begin{cases} \dfrac{m^{n-1}(m^n - 1)}{m-1}\sigma^2 & \text{if } m \neq 1 \\ n\sigma^2 & \text{if } m = 1. \end{cases}$$

## 2.9 References


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