M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Vidyasagar University

Semestar-IV

Paper-MTM 403/II **Paper Name: Probability and Statistics**

Unit 1 Unit Name: **Stochastic Process : Markov Chains**

by

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Unit Structure:

1.1 Introduction

The probability models are more realistic than deterministic models in many problems in different branches of science, humanities, engineering etc. Observations taken at different time points rather than those taken at a fixed period of time began to engage the attention of probabilities. This led to a new concept of indeterminism in dynamic studies. This has been called dynamic indeterminism. Many situations occur in physical and life sciences are studied now not only as a random phenomenon but also as one changing with time or space. Similar considerations are also made in other areas, such as, social sciences, engineering and management and so on. The scope of applications of random variables which are functions of time or space or both has been on the increase.

Objectives:

Gone through this unit the readers will learn the following:

- Stochastic process
- Markov Chain

- Transition probability matrix
- Random walk with absorbing and reflecting barriers
- Probability distribution
- Order of Markov chain
- Markov chain as graphs
- Champman-Kolmogorov equation
- Classification of states and chains
- Closed state
- Irreducible chain
- Persistent and transient states
- Exercise

1.2 Stochastic Process

Families of random variables $\{x(t), t \in T\}$ where T is index set which are functions of say, time are known as stochastic processes (or random processes or random functions).

Example 1.2.1 Consider a simple experiment like throwing a fair die. Suppose that X_n is the outcome of the n^{th} throw, $n \geq 1$. Then $\{X_n, n \geq 1\}$ is a family of random variables such that for a distinct value of $n(= 1, 2, \ldots)$, one gets a distinct random variable X_n; $\{X_n, n \geq 1\}$ constitutes a stochastic process, known as **Bernoulli process**.

Example 1.2.2 Consider a random event occurring in time, such as, number of telephone calls received at a switch board. Suppose that $X(t)$ is the random variable which represents the number of incoming calls in an interval $(0, t)$ of duration t units. The number of calls within a fixed interval of specified duration, say, one unit of time, ia a random variable $X(t)$ and the family $\{X(t), t \in T\}$ constitutes a stochastic process $T = [0, \infty)$.

1.3 Definitions

Let S_t be the sample space of $X(t)$, which may be finite or infinite. The random variables $X_t, X_{t+r}(r > 0)$ may be dependent or independent. Also, T may be finite or infinite then the stochastic process $\{X(t), t \in T\}$ is a **discrete-parameter** stochastic process. If T is any finite or infinite interval *e.g.* $T = \{t : a < t < b\}$, $T = \{t : 0 \leq t < \infty\}$, the process is said to be a **continuous parameter** stochastic process.

The particular value of $X(t)$ is often called **state** and the set of all possible values of a single random variable $X(t)$ of a stochastic process $\{X(t), t \in T\}$ is known as its **state space**.

The system is defined for a continuous range of time and we say that we have a family of random variable in continuous time. A stochastic process in continuous time may have either a discrete or a continuous state space. For example, suppose that $X(t)$ gives the number of incoming calls at a switchboard in an interval $(0, t)$. Here, the state space of $X(t)$ is discrete though $X(t)$ is defined for a continuous range of time.

We have a process in continuous time having a discrete state space. Suppose that $X(t)$ represents the maximum temperature at a particular place in $(0, t)$, then the set of possible values of $X(t)$ is continuous. Here we have a system in continuous time a continuous state space.

We have assumed that the values assumed by the random variable $X(t)$ are one-dimensional, but the process $\{X(t)\}$ may be multi-dimensional. Consider $X(t) = \big(X_1(t), X_2(t)\big)$, where X_1 represents the maximum and X_2 that minimum temperature at a place in an interval of time $(0, t)$. This is a two-dimensional stochastic process in continuous time having continuous state space. One can similarly have multi-dimensional process. One-dimensional processes can be classified, in general, into the following four types of processes:

(i) Discrete time, discrete state space

(ii) Discrete time, continuous state space

(iii) Continuous time, discrete state space

(iv) Continuous time, continuous state space

1.4 Markov Chains

Let us consider a simple coin tossing experiment repeated for a number of times. The possible outcomes at each trial are two: head with probability, say, p and tail with probability $q, p + q = 1$. Let us denote head by 1 and tail by 0 and the random variable denoting the result of the n^{th} toss by X_n. Then for $n = 1, 2, 3 \ldots.$

$$P(X_n = 1) = p,\ P(X_n = 0) = q.$$

Thus we have a sequence of random variables $X_1, X_2 \ldots.$ The trials are independent and the result of the nth does not depend in any way on the previous trials numbered $1, 2, \ldots, n-1$. The random variables are independent.

Consider now the random variable given by the partial sum $S_n = X_1 + X_2 + \ldots + X_n$. The sum S_n gives the accumulated number of heads in the first n trials and its possible values are $0, 1, 2, \ldots, n$.

We have $S_{n+1} = S_n + X_{n+1}$. Given that $S_n = j,\ j = 0, 1, 2 \ldots, n$, the random variables S_{n+1} can assume only to possible values: $S_{n+1} = j$ with probability q and $S_{n+1} = j + 1$ with probability p; these probabilities are not at all affected by the values of the variables $S_1, S_2, \ldots, S_{n-1}$.

Thus

$$P\big(S_{n+1} = j+1/S_n = j\big) = p \text{ and } P\big(S_{n+1} = j/S_n = j\big) = q.$$

This is an example of Markov Chain, a case of simple dependence that the outcome of $(n+1)^{th}$ trial depends directly on that of nth trial and only on it. The conditional probability of S_{n+1} given by S_n depends on the value of S_n and the manner in which the value of S_n was reached is of no consequence.

Now we present the formal definition of Markov Chain.

Def. 1.4.1 *The stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$ is called a Markov Chain, if for $j, k, i_1, i_2 \ldots, i_{n-1} \in N$ (the set of all natural numbers),*

$$P\big(X_n = k/X_{n-1} = j, X_{n-2} = i_1, \ldots, X_0 = i_{n-1}\big) = P\big(X_n = k/X_{n-1} = j\big) = p_{jk} \text{ (say)}, \quad (1.1)$$

whenever the first member is defined.

The out comes are called the states of the Markov Chain. If X_n has outcome j, i.e. $(X_n = j)$, the process is said to be a state j at n^{th} trial. To a pair of states (j, k) at the two successive trials (say, n^{th} and $(n+1)^{th}$ trials) there is an associated conditional probability p_{jk}. It is the probability of transition from the state j at n^{th} trial to the state k at $(n+1)^{th}$ trial. The transition probabilities p_{jk} are basic to the study of the structure of the Markov Chain.

The transition probability may or may not be independent of n. If the transition probability p_{jk} is independent of n, the Markov Chain is said to be **homogeneous** (or to have stationary transition probabilities). If it is dependent on n, the chain is said to be non-homogeneous. The transition probability p_{jk} refers to the state (j,k) at two successive trials (say, n^{th} and $(n+1)^{th}$ trials,) the transition is one step and p_{jk} is called one step or unit step transition probability.

In general case, if the states (j,k) at two non-successive trials, say, state j at the nth trial and state k at the (n+m)th tril. The corresponding transition probability is then called m^{th} step transition probability and is denoted by $p_{jk}^{(m)}$, i.e.,

$$p_{jk}^{(m)} = P(X_{n+m} = k/X_n = j). \quad (1.2)$$

1.4.1 Transition Probability matrix

It may be noted that the transition probabilities p_{jk} satisfy

$$p_{jk} \geq 0, \sum_k p_{jk} = 1 \text{ for all } j. \quad (1.3)$$

These probabilities may be written in the matrix form as

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad (1.4)$$

This matrix is called the **transition probability matrix** or **matrix of transition probabilities (t.p.m)** of the Markov Chain. The matrix P is a stochastic matrix, i.e. a square matrix with non-negative elements and unit row sums.

Example 1.4.1 Random walk with absorbing barriers (Gambler's ruin problem)

A particle performs a random walk with absorbing barriers, say, at 0 and k. Whenever it is at any position $r(0 < r < k)$, it moves to $r + 1$ with probability p or to (r-1) with probability $q, p + q = 1$. But, as soon as it reaches 0 or k it remains there itself. Let X_n be the position of the particle after n moves. The different states of X_n are the different positions of the particle. $\{X_n\}$ is a Markov Chain whose unit-step transition probabilities are given by

$$\text{for } 0 < r < k, \qquad P\big(X_n = r + 1/X_{n-1} = r\big) = p \text{ and } P\big(X_n = r - 1/X_{n-1} = r\big) = q.$$

$$\text{Also} \qquad P\big(X_n = 0/X_{n-1} = 0\big) = 1 \text{ and } P\big(X_n = k/X_{n-1} = k\big) = 1$$

$$\text{and} \qquad P\big(X_n = r/X_{n-1} = r\big) = 0,\ 0 < r < k.$$

The transition matrix is given by

$$\begin{array}{ccc} & & \text{States of} X_n \\ & & 0\ 1\ 2\ 3 \cdots k-2\ k-1\ k \\ & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ k-1 \\ k \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \ldots 0 & 0 & 0 \\ q & 0 & p & 0 \ldots 0 & 0 & 0 \\ 0 & q & 0 & p \ldots 0 & 0 & 0 \\ 0 & 0 & q & 0 \ldots 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \cdots & & \\ 0 & 0 & 0 & 0 \ldots q & 0 & p \\ 0 & 0 & 0 & 0 \ldots 0 & 0 & 1 \end{bmatrix} \end{array}$$

States of X_{n-1}

Table 1.1: Transition probability matrix of Gambler's ruin problem.

Example 1.4.2 Random walk between reflecting barriers

Consider that a particle may be at any position r, $r = 0, 1, 2, \ldots, k(\geq 1)$ of the x-axis. From state r it moves to state $r + 1$, $1 \leq r \leq k - 1$ with probability p and to state $r - 1$ with probability q. As soon as it reaches state 0 it remains there with probability a and is reflected to state 1 with probability $1 - a(0 < a < 1)$; if it reaches state k it remains there with probability b and is reflected to $k - 1$ with probability $1 - b (0 < b < 1)$. Then $\{X_n\}$, where X_n is the position of the particle after n steps or moves, is a Markov Chain with state space $S = \{0, 1, 2, \ldots, k\}$. The transition probabilities are given by

$$\text{for } 0 < r < k, \begin{cases} P(X_n = r+1/X_{n-1} = r) = p, \ P(X_n = r-1/X_{n-1} = r) = q \\ \text{and } \ P(X_n = r/X_{n-1} = r) = 0 \end{cases}.$$

$$\text{Also } P(X_n = 0/X_{n-1} = 0) = a, \ P(X_n = 1/X_{n-1} = 0) = 1 - a$$

$$\text{and } P(X_n = k-1/X_{n-1} = k) = b, \ P(X_n = k/X_{n-1} = k) = 1 - b.$$

The transition matrix is shown below

$$\text{P= States of } X_{n-1} \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ k-1 \\ k \end{array} \begin{array}{c} \text{States of} X_n \\ \begin{array}{ccccccc} 0 & 1 & 2 & 3 \cdots & k-2 & k-1 & k \end{array} \\ \left[\begin{array}{ccccccc} a & 1-a & 0 & 0 \ldots 0 & 0 & 0 \\ q & 0 & p & 0 \ldots 0 & 0 & 0 \\ 0 & q & 0 & p \ldots 0 & 0 & 0 \\ 0 & 0 & q & 0 \ldots 0 & 0 & 0 \\ \ldots & & \ldots & \ldots & & \ldots & \ldots \\ 0 & 0 & 0 & 0 \ldots q & 0 & p \\ 0 & 0 & 0 & 0 \ldots 0 & 1-b & b \end{array} \right] \end{array}$$

Table 1.2: Transition matrix for random walk between reflecting barriers.

Note 1.4.1 *If $a = 1$, then 0 is an absorbing barrier and if $a = 0$ then 0 is a reflecting barrier, if $0 < a < 1$, 0 is an elastic barrier. Similar is the case with state k. The case when both 0 and k are absorbing barriers corresponds to the familiar Gambler's ruin problem (with total capital between the two gamblers is k).*

Example 1.4.3 Partial sum of independent random variables:

Consider a series of coin tossing experiments, where the outcomes of nth trial are denoted by 1 (for a head) and 0 (for a tail). Let X_n be the randon variable denoting the outcome of the nth trial and $S_n = X_1 + X_2 + \ldots + X_n$ be the nth partial sum. The possible values of S_n are $0, 1, 2, \ldots n$, i.e. the states of S_n are $r,\ r=0,1,2,\dots,n,\ \{S_n,\ n\ge 0\}$ is a Markov Chain with transition is given below

$$\text{States of} X_n$$

$$P= \text{States of } X_{n-1} \begin{array}{c} \\ 0\\1\\2\\ \vdots\\ r-1\\ r\\ \vdots\end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2\cdots r-1 & r & \cdots \end{array} \\ \begin{bmatrix} q & p & 0 & \dots & 0 & 0 & \cdots\\ 0 & q & p & \dots & 0 & 0 & \cdots\\ 0 & 0 & q & \dots & 0 & 0 & \cdots\\ \vdots & & \vdots & & \vdots & & \vdots\\ 0 & 0 & 0 & \dots & q & p & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\end{array}$$

Mathematically,

$$p_{jk} = P(S_n = k/S_{n-1} = j) = \begin{cases} p, & k = j+1\\ q, & k = j\\ 0, & \text{otherwise.}\end{cases}$$

1.4.2 Probability Distribution

The probability distribution of the random variables $X_r, X_{r+1}, \dots, X_{r+n}$ can be computed in terms of the transition probabilities p_{jk} and the initial distribution of X_r. Suppose $r = 0$, then

$$\begin{aligned} &P(X_0 = a, X_1 = b, \dots, X_{n-2} = i, X_{n-1} = j, X_n = k)\\ &= P(X_n = k/X_{n-1} = j, \dots, X_0 = a)P(X_{n-1} = j, \dots, X_0 = a)\\ &= P(X_n = k/X_{n-1} = j)P(X_{n-1} = j, \dots, X_0 = a) \text{ Since } \{X_n\} \text{ is a Markov Chain}\\ &= P(X_n = k/X_{n-1} = j)P(X_{n-1} = j/X_{n-2} = i)P(X_{n-2} = i, \dots, X_0 = a)\\ &= P(X_n = k/X_{n-1} = j)P(X_{n-1} = j/X_{n-2} = i)\dots P(X_1 = b/X_0 = a)P(X_0 = a)\\ &= P(X_0 = a)p_{ab}\dots p_{ij}p_{jk}.\end{aligned}$$

Thus

$$\begin{aligned} &P(X_r = a, X_{r+1} = b, \dots, X_{r+n-2} = i, X_{r+n-1} = j, X_{r+n} = k)\\ &= P(X_r = a)\ p_{ab}\dots p_{ij}\ p_{jk}.\end{aligned}$$

Example 1.4.4

Let $\{X_n, n \geq 0\}$ be a Markov Chain with three states 0,1,2 and with transition matrix

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

and the initial distribution $P(X_0 = i) = \frac{1}{3},\ i = 0, 1, 2.$
We have

$$P(X_1 = 1/X_0 = 2) = \frac{3}{4}, \quad P(X_2 = 2/X_1 = 1) = \frac{1}{4}$$

$$\begin{aligned} P(X_2 = 2, X_1 = 1/X_0 = 2) &= P(X_2 = 2/X_1 = 1)P(X_2 = 1/X_0 = 2) \\ &= \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}. \end{aligned}$$

$$\begin{aligned} P(X_2 = 2, X_1 = 1, X_0 = 2) &= P(X_2 = 2, X_1 = 1/X_0 = 2)P(X_0 = 2) \\ &= \frac{3}{16} \cdot \frac{1}{3} = \frac{1}{16}. \end{aligned}$$

$$\begin{aligned} P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2) &= P(X_3 = 1/X_2 = 2, X_1 = 1, X_0 = 2)P(X_2 = 2, X_1 = 1, X_0 = 2) \\ &= P(X_3 = 1/X_2 = 2)\frac{1}{16} = \frac{3}{4} \cdot \frac{1}{16} = \frac{3}{64}. \end{aligned}$$

Note 1.4.2 *The matrix of transition probabilities together with the initial distribution completely specifies a Markov Chain* $\{X_n,\ n = 0, 1, 2, \ldots\}$.

Theorem 1.1 (General existence theorem of Markov Chains) *Given the set* N *and the sequence of stochastic matrices* ${}^{(n)}P$*, there exists a Markov Chain* $\{X_n,\ n \geq 0\}$ *with state space* N *and transition probability matrix* ${}^{(n)}P$.

1.4.3 Strong Markov propertry

Let N be a stopping time (is also a random variable) for a Markov Chain $\{X_n,\ n > 0\}$ and let A and B be two events relating to X_n and happening, prior and posterior respectively to N. Then

$$P(B/X_N = i, A) = P(B/X_N = i). \tag{1.5}$$

This is called the strong Markov property. It shows that if N is a stopping time for a Markov Chain $\{X_n,\ n > 0\}$, then the evolution of the chain starts afresh from the state reached at time N.

Every discrete time Markov Chain $\{X_n,\ n \geq 0\}$ possesses the strong Markov property.

1.4.4 Order of Markov Chain

A Markov Chain $\{X_n\}$ is said to be of order $s(s = 1, 2, 3 \ldots)$ if for all n,

$$\begin{aligned} &P(X_n = k/X_{n-1} = j, X_{n-2} = j_1, \ldots, X_{n-s} = j_{s-1} \ldots) \\ &= P(X_n = k/X_{n-1} = j, \ldots X_{n-s} = j_{s-1}) \end{aligned} \tag{1.6}$$

whenever L.H.S is defined.

A Markov Chain $\{X_n\}$ is said to be of order one (or simply a Markov Chain) if

$$\begin{aligned} &P(X_n = k/X_{n-1} = j, X_{n-2} = j_1, \ldots) \\ &= P(X_n = k/X_{n-1} = j) = p_{jk} \end{aligned}$$

whenever $P(X_{n-1} = j, X_{n-2} = j_i, \ldots) > 0$.

Unless explicitly stated otherwise, we shall mean by Markov Chain, a chain of order one. A chain is said to be of order zero if $p_{ik} = p_k$ for all j. This implies independence of X_n and X_{n-1}.

1.4.5 Markov Chains as graphs

The states of a Markov Chain may be represented by the vertices of a graph and one step transitions between states by directed arcs, if $i \to j$, then vertices i and j are joined by a directed arc with arrow towards j, the value of p_{ij}which is the weight of the directed arc (i, j). If $V = (1, 2, \ldots m)$ is the set of vertices corresponding the state space of the chain and E is the set of directed arcs between these vertices, then the graph $G = (V, E)$ is the directed graph or digraph or transition graph of the chain. A digraph such that its arc weights are positive and sum of the arc weights of the arc from each node is unity is called a stochastic graph. The digraph or transition graph of a Markov Chain is a stochastic graph.

A transition graph is a great aid in visualizing a Markov chain, it is an useful tool in studying the properties of the chain, e.g., the graph of the Chain of the transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array} \tag{1.7}$$

is

1.5 Higher Transition Probabilities

The m-step transition probability is denoted by

$$P(X_{m+n} = k/X_n = j) = p_{jk}^{(m)}. \tag{1.8}$$

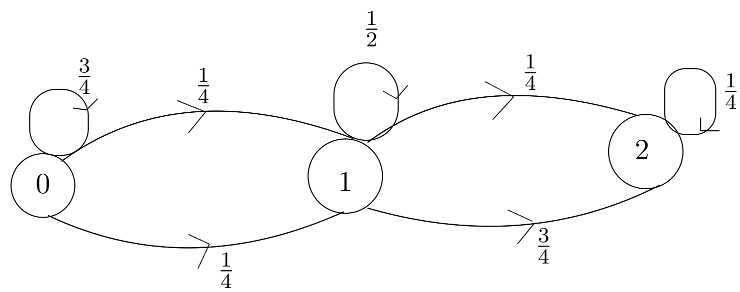

Figure 1.1: Transition graph of the Markov Chain of transition matrix (1.13)

$p_{jk}^{(m)}$ gives the probability that from the state j at nth trial, the state k is reached at $(m+n)^{th}$ trial in m steps, i.e. the probability of transition from the state j to the state k in exactly m steps. The number m does not occur in the R.H.S of (1.8) and the chain is called homogeneous otherwise it is called non-homogeneous.

Chapman-Kolmogorov Equation:

Let $p_{jk}^{(m)}$ be the m-step transition probability from state j to the state k. Then

$$p_{jk}^{(m+n)} = \sum_r p_{rk}^{(n)} p_{jr}^{(m)} = \sum_r p_{jr}^{(n)} p_{rk}^{(m)} \tag{1.9}$$

$$\text{i.e.}\quad P^{m+n} = P^n P^m = P^m P^n \quad \text{where } P^n = \left(p_{ij}^{(n)}\right). \tag{1.10}$$

Proof. the m-step transition probability is defined as

$$P(X_{m+n} = k/X_n = j) = p_{jk}^{(m)}.$$

The one step transition probabilities $p_{jk}^{(1)}$ are denoted by p_{jk}. The 2-step transition probability $p_{jk}^{(2)}$ is given by

$$p_{jk}^{(2)} = P(X_{n+2} = k/X_n = j).$$

The state k can be reached from the state j in two steps through some intermediate state r. Consider a fixed value of r, we have

$$\begin{aligned} &P(X_{n+2} = k, X_{n+1} = r/X_n = j) \\ &= P(X_{n+2} = k/X_{n+1} = r, X_n = j)P(X_{n+1} = r/X_n = j) \\ &= p_{rk}^{(1)} p_{jr}^{(1)} = p_{jr} p_{rk}. \end{aligned}$$

Since these intermediate states r can assume the values $r = 1, 2, \ldots$, we have

$$\begin{aligned} p_{jk}^{(2)} = P(X_{n+2} = k/X_n = j) &= \sum_r P(X_{n+2} = k, X_{n+1} = r/X_n = j) \\ &= \sum_r p_{jr} p_{rk} (\text{summing over for all the intermediate states}) \end{aligned}$$

By similar method,

$$\begin{aligned} p_{jk}^{(m+1)} &= P(X_{m+n+1} = k/X_n = j) \\ &= \sum_r P(X_{m+n+1} = k/X_{n+m} = r)P(X_{n+m} = r/X_n = j) \\ &= \sum_r p_{rk} p_{jr}^{(m)}. \end{aligned}$$

Similarly, we get

$$p_{jk}^{(m+1)} = \sum_r p_{jr} p_{rk}^{(m)}.$$

In general, we have

$$p_{jk}^{(m+n)} = \sum_r p_{rk}^{(n)} p_{jr}^{(m)} = \sum_r p_{jr}^{(n)} p_{rk}^{(m)}. \tag{1.11}$$

Denoting by $P^n = (p_{ij}^{(n)})$ the matrix of n-step transition probabilities, the equation (1.11) can be written as

$$P^{m+n} = P^n P^m. \tag{1.12}$$

The equations (1.11) and (1.12) are called Chapman-Kolmogorov equation and these characterize Markov Chains.

Example 1.5.1

Suppose that probability of a dry day (state 0) following a rainy day (state 1) is $\frac{1}{3}$ and that the probability of a rainy day following a dry day is $\frac{1}{2}$ and transition probability matrix is

$$p = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

If May 10 is a dry day then what are the probabilities that May 12 and May 14 are dry days?

Solution. We have $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

Then $P^2 = P \cdot P = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{bmatrix}$ and $P^4 = \begin{bmatrix} \frac{173}{432} & \frac{259}{432} \\ \frac{259}{648} & \frac{389}{648} \end{bmatrix}$

If May 10 is a dry day, the probability that May 12 is a dry day is $p_{00}^2 = \frac{5}{12}$ and that May 14 a dry is $p_{00}^{(4)} = 173/432$.

Example 1.5.2 Consider a communication system which transmits the two digits 0 and 1 through several stages. Let $X_n,\ n \geq 1$ be the digit leaving the n^{th} stage of system and X_0 be the digit entering the first stage (leaving the 0th stage). At each stage there is a constant probability q that the digits which enters will be transmitted unchanged (i.e. the digit will remain unchanged when it leaves), and probability p otherwise (i.e. the digit changes when it leaves), $p + q = 1$. Find the one step transition matrix P, and n-step transition matrix P^n. Also find P^n when $n \to \infty$.

Solution. Here $\Big\{X_n,\ n \geq 0\Big\}$ is a homogeneous two-state Markov Chain with unit-step transition matrix $P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$.

$$P = \begin{pmatrix} q & p \\ p & q \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(p+q) + \frac{1}{2}(q-p) & \frac{1}{2}(p+q) - \frac{1}{2}(q-p) \\ \frac{1}{2}(p+q) - \frac{1}{2}(q-p) & \frac{1}{2}(p+q) + \frac{1}{2}(q-p) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p) & \frac{1}{2} - \frac{1}{2}(q-p) \\ \frac{1}{2} - \frac{1}{2}(q-p) & \frac{1}{2} + \frac{1}{2}(q-p) \end{pmatrix}$$

$$\text{Now,} \quad P^2 = \begin{pmatrix} q & p \\ p & q \end{pmatrix} \begin{pmatrix} q & p \\ p & q \end{pmatrix} = \begin{pmatrix} p^2 + q^2 & 2pq \\ 2pq & p^2 + q^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(q+p)^2 + \frac{1}{2}(q-p)^2 & \frac{1}{2}(q+p)^2 - \frac{1}{2}(q-p)^2 \\ \frac{1}{2}(q+p)^2 - \frac{1}{2}(q-p)^2 & \frac{1}{2}(q+p)^2 + \frac{1}{2}(q-p)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p)^2 & \frac{1}{2} - \frac{1}{2}(q-p)^2 \\ \frac{1}{2} - \frac{1}{2}(q-p)^2 & \frac{1}{2} + \frac{1}{2}(q-p)^2 \end{pmatrix}$$

$$\text{Let} \quad P^m = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p)^m & \frac{1}{2} - \frac{1}{2}(q-p)^m \\ \frac{1}{2} - \frac{1}{2}(q-p)^m & \frac{1}{2} + \frac{1}{2}(q-p)^m \end{pmatrix}$$

$$\text{Now,} \quad P^{m+1} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p)^m & \frac{1}{2} - \frac{1}{2}(q-p)^m \\ \frac{1}{2} - \frac{1}{2}(q-p)^m & \frac{1}{2} + \frac{1}{2}(q-p)^m \end{pmatrix} \begin{pmatrix} q\ p \\ p\ q \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p)^{m+1} & \frac{1}{2} - \frac{1}{2}(q-p)^{m+1} \\ \frac{1}{2} - \frac{1}{2}(q-p)^{m+1} & \frac{1}{2} + \frac{1}{2}(q-p)^{m+1} \end{pmatrix}.$$

This is true for $n = m + 1$. Hence by mathematical induction

$$P^n = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p)^n & \frac{1}{2} - \frac{1}{2}(q-p)^n \\ \frac{1}{2} - \frac{1}{2}(q-p)^n & \frac{1}{2} + \frac{1}{2}(q-p)^n \end{pmatrix}$$

Here $p_{00}^{(n)} = p_{11}^{(n)} = \frac{1}{2} + \frac{1}{2}(q-p)^n$ and $p_{10}^{(n)} = p_{01}^{(n)} = \frac{1}{2} - \frac{1}{2}(q-p)^n$.

As $n \to \infty,\ p_{00}^{(n)} = p_{11}^{(n)} = \frac{1}{2}$ and $p_{10}^{(n)} = p_{01}^{(n)} = \frac{1}{2}$ as $q - p < 1$.

1.6 Classification of States and Chains

1.6.1 Communication relations

If $p_{ij}^{(n)} > 0$ for some $n \geq 1$, then we say that j can be reached or state j is accessible from state i. This relation is denoted by $i \to j$. Conversely, if for all n, $p_{ij}^{(n)} = 0$, then j is not accessible from i and it is denoted by $i \nrightarrow j$.

If two states i and j are such that each is accessible from the other then we say that the two states communicate and it is denoted by $i \leftrightarrow j$. Then there exists integers $m(\geq 1)$ and $n(\geq 1)$ such that $p_{ij}^{(n)} > 0$ and $p_{ij}^{(m)} > 0$. Obviously, the relation $\leftrightarrow$ is symmetric.

Theorem 1.2 *The communicate relation $\leftrightarrow$ is transitive.*

Proof. From Chapman-Kolmogorov equation

$$p_{ik}^{(m+n)} = \sum_r p_{ir}^{(m)} p_{rk}^{(n)}$$
$$\text{or, } p_{ik}^{(m+n)} > p_{ij}^{(m)} p_{jk}^{(n)}.$$

If $i \to j$ and $j \to k$ then $p_{ij}^{(m)} > 0$ and $p_{jk}^{(n)} > 0$.
Then $p_{ik}^{(m+n)} > 0$ implies $i \to k$.
Again $p_{ki}^{(m+n)} \geq p_{kj}^{(m)} p_{ji}^{(n)}$.
If $k \to j$ and $j \to i$ then by similar way $p_{ki}^{(m+n)} > 0$.
Hence $i \leftrightarrow j$ and $j \leftrightarrow k$ implies $i \leftrightarrow k$, i.e., $\leftrightarrow$ is transitive.

1.6.2 Class property

A class of states is a subset of the state space such that every states of the class communicates with every other and there is no other state outside the class which communicates with all other states in the class. A property defined for all states of a chain is a class property if its possession by one state in a class implies its possession by all states of the same class.

The state i is a return state if $p_{ii}^{(n)} > 0$ for sum $n \geq 1$.

The period d_i of a return to state i is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$ i.e.,

$$d_i = \text{ G.C.D } \{m : \ p_{ii}^{(m)} > 0\}.$$

[G.C.D means greatest common divisor.]

The state i is said to be **aperiodic** if $d_i = 1$ and **periodic** if $d_i > 1$. Clearly, state i is periodic if $p_{ii} \neq 0$.

Closed state: If C is a set of states such that no state outside C can be reached from any sate in C, then C is said to be closed.

If C is closed and $j \in C$ while k not belongs to C, then $p_{jk}^{(n)} = 0$ for all n, i.e. C is closed iff $\sum_{j \in C} p_{ij} = 1$ for every $i \in C$. A closed state may contain one or more states. If a closed state contains only one state j then state j is said to be absorbing i.e. j is absorbing iff $p_{jj} = 1, p_{jk} = 0, k \neq j$.

In Gambler's ruin problem states 0 and k are absorbing.

Every finite Markov Chain contains at least one closed set i.e. the set of all states or the state space.

Irreducible Chain: If the chain does not contain any other proper closed subset other than the state space, then the chain is called irreducible; the t.p.m of irreducible chain is an irreducible matrix.

In an irreducible Markov Chain every state can be reached form every other state. Chains which are not irreducible are said to be reducible or non-reducible, the t.p.m is reducible.

Notations: Let $f_{jk}^{(n)}$ be the probability that it reaches from the state j to the state k for the first time at the nth step (or after n transitions) and let $p_{jk}^{(n)}$ be the probability that it reaches state k (not necessarily for the first time) after n transitions.

Let F_{jk} denotes the probability that starting with state j the system will ever reach state k.

$$\text{That is,}\ \ F_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}. \tag{1.13}$$

The mean time from state j to state k is given by

$$\mu_{jk} = \sum_{n=1}^{\infty} n f_{jk}^{(n)}. \tag{1.14}$$

The mean recurrence time for the state j is

$$\mu_{jj} = \sum_{n=1}^{\infty} f_{jj}^{(n)} \tag{1.15}$$

Theorem 1.3 (First Entrance Theorem)

For any state j and k

$$p_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)},\ n \geq 1 \tag{1.16}$$

$$\textit{with}\ \ p_{kk}^{(0)} = 1,\ f_{jk}^{(0)} = 0,\ f_{jk}^{(1)} = p_{jk}.$$

Proof. We have

$$
\begin{aligned}
p_{jk}^{(n)} &= \text{ probability that the system will pass from state j to state k in n steps} \\
&= \sum_{r=0}^{n} P(\text{first return to k occurs at } r^th \text{ step from state j}) \\
&\times P(\text{the system is in k at } n^th \text{ step/it was in state k at the } r^th \text{ step}) \\
&= \sum_{r=0}^{n} f_{jk}^{(r)} P(X_n = k/X_r = k) \\
&= \sum_{r=0}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}.
\end{aligned}
$$

Therefore, $p_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}$.

Note 1.6.1 *The above result can also be written as*

$$
\begin{aligned}
p_{jk}^{(n)} &= \sum_{r=0}^{n-1} f_{jk}^{(r)} p_{kk}^{(n-r)} + f_{jk}^{(n)} [\because p_{kk}^{(0)} = 1] \\
or, \quad f_{jk}^{(n)} &= p_{jk}^{(n)} - \sum_{r=0}^{n-1} f_{jk}^{(r)} p_{kk}^{(n-r)}. \qquad (1.17)
\end{aligned}
$$

1.6.3 Transient and Persistent (recurrent) states

A state j is said to be persistent (or recurrent) if return to j is certain, i.e, $F_{ij} = 1$ (i.e., return to state j is certain).

A state j is called transient (or non-recurrent) if return to j is uncertain, i.e., $F_{ij} < 1$.

A persistent state j is said to be null-persistent if $\mu_{jj} = \infty$, i.e., if the mean recurrence time is infinite, and is said to be non-null (or positive) persistent if $\mu_{jj} < \infty$.

A persistent state j is called ergodic if it is not a null-state and is non-periodic.

Note 1.6.2 *If the state j is transient then* $p_{jj}^{(n)} \to 0$ *an* $n \to \infty$ *because the return to j is uncertain.*

Example 1.6.1 Let $\{X_n,\ n \geq 1\}$ be a Markov chain having state space S={1,2,3,4} and transition matrix $P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Identify the states as transient, persistent, ergodic.

Solution. Here $f_{11}^{(1)} = \frac{1}{3},\ f_{11}^{(2)} = \frac{2}{3}$,

$f_{11}^{(1)} = 0,\ n \geq 3$ and $F_{11} = \frac{1}{3} + \frac{2}{3} = 1$. So that state 1 is persistent.

$f_{33}^{(1)} = \frac{1}{2},\ f_{33}^{(n)} = 0,\ n \geq 2$.

So $F_{33} = \frac{1}{2} < 1$. Therefore, the state 3 is transient.

$f_{44}^{(1)} = \frac{1}{2},\ f_{44}^{(n)} = 0,\ n \geq 2$ so that $F_{44} = \frac{1}{2} < 1$. Thus state 4 is also transient.

Further, $\mu_{11} = 1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{5}{3}$. The state is non-null persistent. Again $p_{11} = \frac{1}{3} > 0$, so that state 1 is aperiodic, state 1 is ergodic.

$$f_{22}^{(1)} = 0,\ f_{22}^{(2)} = 1 \cdot \frac{2}{3},\ f_{22}^{(3)} = 1 \cdot \frac{1}{3} \cdot \frac{2}{3},\ f_{22}^{(4)} = 1 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} \quad \ldots, f_{22}^{(n)} = 1 \cdot \left(\frac{1}{3}\right)^{n-2} \cdot \frac{2}{3},\ n \geq 2.$$

$$\therefore \quad F_{22} = \sum_{n=1}^{\infty} f_{22}^{(n)} = \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^{n-2} \cdot \frac{2}{3} = 1.$$

Thus state 2 is persistent.

$$\text{Now, } \mu_{22} = \sum_{k=1}^{\infty} k f_{22}^{(k)} = \sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^{k-2} \cdot \frac{2}{3} = 2 \sum_{k=2}^{\infty} k \left(\frac{1}{3}\right)^{k-1} = \frac{5}{2}.$$

Therefore, state 2 is non-null persistent. It is also aperiodic and hence ergodic.

Theorem 1.4 *In an irreducible chain, all the states are of the same type. They are either all transient, all persistent null or all persistent non-null. All the states are aperiodic and in the later case they all have the same period.*

Proof. Since the chain is irreducible, every state can be reached from every other state. If i, j are any two states then i can be reached from j and j from i, i.e.,

$$p_{ij}^{(N)} = \alpha > 0 \ \text{ for some } N \geq 1 \text{ and } p_{ji}^{(M)} = \beta > 0 \ \text{ for some } M \geq 1.$$

We have

$$p_{jk}^{(m+n)} = \sum_r p_{jr}^{(m)} p_{rk}^{(n)} \geq p_{jr}^{(m)} p_{rk}^{(n)} \text{ for each } r.$$

Hence

$$p_{ii}^{(n+N+M)} \geq p_{ij}^{(N)} p_{jj}^{(n)} p_{ji}^{(M)} = \alpha\beta p_{jj}^{(n)} \tag{1.18}$$

$$\text{and} \quad p_{jj}^{(n+N+M)} \geq p_{ji}^{(M)} p_{ii}^{(n)} p_{ij}^{(N)} = \alpha\beta p_{ii}^{(n)}. \tag{1.19}$$

From above it is clear that the two series $\sum\limits_n p_{ii}^{(n)}$ and $\sum\limits_n p_{jj}^{(n)}$ converge or diverge together. Thus the two states i,j are either both transient or both persistent.

Suppose that i is persistent null, then $p_{ii}^{(n)} \to 0$ as $n \to \infty$, then from (1.18), $p_{jj}^{(n)} \to 0$ as $n \to \infty$, so that j is also persistent null, i.e., they are both persistent null.

Suppose that i is persistent non-null and has period t, then $p_{ii}^{(n)} > 0$ whenever n is a multiple of t. Now

$$p_{ii}^{(N+M)} \geq p_{ij}^{(N)} p_{ji}^{(M)} = \alpha\beta > 0,$$

so that $(N + M)$ is a multiple of t. From (1.19)

$$p_{jj}^{(n+N+M)} \geq \alpha\beta p_{ii}^{(n)} > 0.$$

Thus $(n + N + M)$ is a multiple of t and so is the period of the state j also.

Theorem 1.5 *State j is persistent iff*

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty \tag{1.20}$$

.

Proof. Let

$$P_{jj}(s) = \sum_{n=0}^{\infty} p_{jj}^{(n)} s^n = 1 + \sum_{n=1}^{\infty} p_{jj}^{(n)} s^n, \ |s| \leq 1$$

and

$$F_{jj}(s) = \sum_{n=0}^{\infty} f_{jj}^{(n)} s^n = \sum_{n=1}^{\infty} f_{jj}^{(n)} s^n, \ |s| \leq 1$$

[$\because$ $p_{jj}^{(0)} = 1$ and $f_{jj}^{(0)} = 0$] be the generating functions of the sequences $\{p_{jj}^{(n)}\}$ and $\{f_{jj}^{(n)}\}$ respectively.

We know that

$$p_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} p_{jj}^{(n-r)}. \tag{1.21}$$

Multiplying both sides by s^n and adding for all $n \geq 1$, we get

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} s^n = \sum_{n=1}^{\infty} \sum_{r=0}^{n} f_{jj}^{(r)} p_{jj}^{(n-r)} s^n$$

$$\text{or,} \quad P_{jj}(s) - 1 = F_{jj}(s) P_{jj}(s). \tag{1.22}$$

The R.H.S is immediately obtained by considering the fact that the R.H.S of (1.21) is a convolution of $\{f_{jj}\}$ and $\{p_{jj}\}$ and that the generating function of the convolution is the product of the two generating functions. Thus we have

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)}, \ |s| \leq 1. \tag{1.23}$$

If the state j is persistent then $F_{jj} = 1$.

$$\therefore \lim_{s\to 1} F_{jj}(s) = 1.$$

Thus $\lim\limits_{s\to 1} P_{jj}(s) \to \infty$ or $\sum\limits_{n=0}^{\infty} p_{jj}^{(n)} = \infty$.

Conversely, if the state is transient then

$$F_{jj} < 1, \; i.e., \; \lim_{s\to 1} F_{jj}(s) < 1$$

$$\text{or,} \quad \lim_{s\to 1} P_{jj}(s) < \infty, \sum_{n=0}^{\infty} p_{jj}^{(n)} < \infty.$$

Remarks:

(i) State j is transient if $\sum\limits_{n=0}^{\infty} p_{jj}^{(n)} < \infty$; this implies that if j is transient then $p_{jj}^{(n)} \to \infty$ as $n \to \infty$.

(ii) The state space of a finite Markov Chain must contain at least one persistent state.

(iii) If k is transient state and j is an arbitrary state then $\sum\limits_{n=0}^{\infty} p_{jk}^{(n)}$ converges and $\lim\limits_{n\to\infty} p_{jk}^{(n)} \to 0$.

(iv) If a Markov Chain having a set of transient states T, starts in a transient state, then with probability 1, it stays at the transient set of states T only a finite number of times after which it enters a recurrent state where it remains forever.

Example 1.6.2 Consider the Markov Chain with t.p.m

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Test whether the states are periodic and persistent.

Solution. The chain is irreducible as the matrix P is irreducible.

We have $P^2 = P \cdot P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$, $P^3 = P$. In general, $P^{2n} = P^2, \; P^{2n+1} = P$.

Therefore, $p_{ii}^{(2n)} > 0, \; p_{ii}^{(2n+1)} = 0$ for all i.

The states are periodic with period 2.

Also, $f_{11}^{(1)} = 0, f_{11}^{(2)} = 1, \; F_{11} = \sum\limits_{n=1}^{\infty} f_{11}^{(n)} = 1$, i.e., state 1 is persistent and hence the other states 0 and 2 are also persistent.

Now, $\mu_{11} = \sum\limits_{n=1}^{\infty} n f_{11}^{(n)} = 2$,

i.e., state 1 is non-null. Thus the states of the chain are periodic and persistent non-null.

1.7 Markov Chains with Continuous State Space

We have discussed Markov Chains $\{X_n\ ,\ = 0, 1, 2, \ldots\}$ with discrete state space, i.e., with $0, \pm 1, \pm 2, \ldots$ as possible values of X_n. Here we consider chains $\{X_n\}$ with continuous state space, i.e., with $(-\infty, \infty)$ as possible range of values of X_n. We shall have to use either probability distribution(d.f.) or probability density function (p.d.f), when this exists, in place of probability mass functions.

Def. 1.7.1 *If for all m and for all possible values of X_m in $(-\infty, \infty)$*

$$\begin{aligned} & P(X_{m+1} \leq x/X_m = y, X_{m-1} = y_1, \ldots, X_0 = y_m) \\ & = P(X_{m+1} \leq x/X_m = y) \end{aligned} \tag{1.24}$$

then $\{X_n, m > 0\}$ is said to constitute a Markov Chain with continuous state space. If the conditional d.f. as given by (1.24) is independent of m, then the chain is homogeneous and equation (1.24) gives one-step transition probability d.f. More generally, the n-step transition probability d.f. is defined by

$$\begin{aligned} & P(X_{m+1} \leq x/X_m = y, X_{m-1} = y_1, \ldots, X_0 = y_m) \\ & = P(X_{m+1} \leq x/X_m = y) \\ & = P_n(y; x). \end{aligned} \tag{1.25}$$

Denote $P(y; x) = P_1(y; x) = P(X_{m+1} \leq x/X_m = y) = P(X_{n+1} \leq x/X_n = y)$.

Let $P_n(x) = P(X_n \leq x)$. The transition d.f. $P_n(y; x)$ and the initial distribution $P(X_0 \leq x) = P_0(x)$ can uniquely determine $P_n(x)$. The Chapman-Kolmogorov equation takes the form

$$P_{n+m}(y; x) = \int_{-\infty}^{\infty} P_n(y; z) P_m(z; x) dz,\ m, n \geq 0 \tag{1.26}$$

which corresponds to $p_{jk}^{(n+m)} = \sum_s p_{js}^{(n)} p_{sk}^{(m)}$ for Markov Chains with discrete state space.

For $n = 1$, equation (1.26) becomes

$$P_{n+1}(y; x) = \int P_n(y; z) P_1(z; x) dz = \int P_n(y; z) P(z; x) dz \tag{1.27}$$

Suppose that as $n \to \infty, P_n(y; x)$ tends to a limit $P(x)$ independent of the initial value.

Then the limiting distribution P(x) satisfies the integral equation

$$P(x) = \int P(z) P(z; x) dz = \int P(z; x) dP(z) \tag{1.28}$$

In the above relations, distribution functions can be replaced by p.d.f's when these exist.

1.8 Unit Summary

The stochastic process is introduced in this unit. The notion of Markov Chain and the transition probability matrix is defined. Several properties Markov Chain are also discussed here. It has been shown that the Markov Chain can be represented by digraph. Different types of states and Chains are defined with examples. Communication relation between two states is defined here. Few illustrative examples are carried out in this chapter. List of exercises are appended alon with this chapter.

1.9 Self Assessment Questions

1. Define stochastic process with example. Classify it with respect to state space and time.
2. Define Markov Chain with example. What do you mean by state and transition probability? What do you mean by transition matrix?
3. State Gambler's ruin problem and write transition matrix for it.
4. Write transition matrix for the problem of random walk between reflecting barriers.
5. Define order of a Markov Chain. Discuss how a Markov Chain can be represented as a graph.
6. State and prove Chapman-Kolmogorov equation.
7. Suppose that probability of a dry day following a rainy day is $\frac{2}{3}$ and that the probability of a rainy day following a dry day is $\frac{1}{2}$ and t.p.m. $P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. If June 2 is a dry day then find the probability that June 4 and June 6 are dry day.
8. Define the following:
   (a) accessible state
   (b) return state
   (c) periodic state
   (d) aperiodic state
   (e) closed state
   (f) irreducible chain
   (g) persistent state
   (h) transient state and
   (i) ergodic

9. State and prove first entrance theorem.

10. Prove that the state j is persistent iff $\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty$.

1.10 References

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